Class number one problem for real quadratic fields of a certain type

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Introduction

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- Class group = free group of fractional ideals/principal fractional ideals
- Class number $h(d)$ = the finite order of the class group

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  2. There are infinitely many $d > 0$, for which $h(d) = 1$. (open)

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  Real quadratic fields are harder to deal with it.

Introduction

Dirichlet class number formula

$$h(d) = \frac{\omega}{2\pi} |d|^{1/2} L(1, \chi_d), \quad d < 0,$$

$$h(d) \log \epsilon_d = d^{1/2} L(1, \chi_d), \quad d > 0,$$

where $\chi_d = \left(\frac{\cdot}{d}\right)$ is the Jacobi symbol, ω is the number of roots of unity in K and ϵ_d is the fundamental unit of $\mathbb{Q}(\sqrt{d})$ for $d > 0$.

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$L(1,\chi_d) \gg_\epsilon |d|^{-\epsilon}$.

If ϵ_d is small, i.e. $\log\epsilon_d \asymp \log d$, then $h(d) \gg_\epsilon d^{1/2-\epsilon}$ and $h(d) \to \infty$ (just like for $d < 0$).

Richaud-Degert (R-D) discriminants:

$$d = (an)^2 + ka \text{ with } a, n > 0\,,\ \pm k \in \{1, 2, 4\}\,.$$

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- Recall that Siegel's theorem is ineffective.
- Class number one problem : Find the exact d for which $h(d) = 1$.

Class Number One Problem for R-D Fields

Biró solves the class number one problem in the following cases:

Theorem (Biró 2003)

- *Yokoi's conjecture is true : Let* $d = n^2 + 4$*. Then* $h(d) > 1$ *if* $n > 17$*;*
- *Chowla's conjecture is true : Let* $d = 4n^2 + 1$*. Then* $h(d) > 1$ *if* $n > 13$*.*

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Until now not known results for two-parameter R-D discriminants without GRH, except

Theorem (L.,2012)

If $d = (an)^2 + 4a$ *is square-free for the odd positive integers* a *and* n *and* $43 \cdot 181 \cdot 353$ *divides* n*, then* $h(d) > 1$*.*

Class Number One Problem for R-D Fields

Theorem (Biró, Gyarmati, L., 2014)

If $d = (an)^2 + 4a$ *is square-free for* a *and* n *odd positive integers and* $d > 1253$, *then* $h(d) > 1$.

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Theorem (Biró, Gyarmati, L., 2014)

If $d = (an)^2 + 4a$ is square-free for a and n odd positive integers and $d > 1253$, then $h(d) > 1$.

Tools we use to prove the theorem:

- In the R-D fields "small primes are inert".
- Formula for a special value of a "sectorial" Dedekind zeta function (after Biró and Granville).
- Computer calculations.
- If $(43 \cdot 181 \cdot 353) \mid n$, then $h\left((an)^2 + 4a\right) > 1$.

Proof

From now on assume that $h(d) = 1$ for the square-free discriminant $d = (an)^2 + 4a$, and $a > 1$.

Small primes are inert

We have that a and $an^2 + 4$ are primes, and for any prime $p \neq a$ such that $2 < p < an/2$ we have

$$\left(\frac{d}{p}\right) = -1\,.$$

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The Condtion $q \longrightarrow r$

- χ is an odd primitive character with conductor $q > 1$ and $(q, 2d) = 1$.
- The ideal $\mathfrak{R} \in \mathbb{Z}[\zeta_q]$ over the odd prime r is such that

$$\sum_{1 \leq u \leq q-1} u\chi(u) \in \mathfrak{R}.$$

Consider the 2-dimensional "Gauss sum"

$$G_\chi(a,n) = \sum_{1\le u,v\le q-1} \chi(au^2 + anuv - v^2)uv\,.$$

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Main identity

If $q \to r$ holds and $h(d) = 1$, then

$$4G_\chi(a,n) + n\,(a + \bar{\chi}(a))\,B \equiv 0 \pmod{\mathfrak{R}}$$

for a certain $B \in \mathbb{Z}[\zeta_q]$.

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- If $q \to r$ holds, the main identity "sieves" the couples (a,n) (mod qr).
- We check with computer if the main identity holds for suitably chosen q and r.

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- Many different parameters q and r, e.g.

$$5 \times 19 \to 13,$$

$$7 \times 19 \to 13, 37, 73,$$

$$13 \times 19 \to 3, 7, 73, 127,$$

$$181 \to 5, 37\,.$$

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if $an > 2 \cdot 3315$(Jacobi symbol condition).

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  if $an > 2 \cdot 3315$(Jacobi symbol condition).
- This contradicts Theorem L. Therefore $an < 2.3315$.

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Theorem (L., 2012)

Let $n = m(m^2 - 3 \cdot 136)$ *for a positive odd integer* m*, and* $N = 2^2 \cdot 3^3 \cdot 7 \cdot 43 \cdot 61 \cdot 137$*. If* $d = n^2 + 4$ *is square-free and* $\left(\frac{d}{N}\right) = -1$*, then for every* $\epsilon > 0$ *there exists an effective computable constant* $c_\epsilon > 0$*, depending only on* ϵ*, such that*

$$h(d) = h(n^2 + 4) > c_\epsilon \left(\log d\right)^{1-\epsilon} .$$

Effective Lower Bounds for $h(d)$

Theorem (Goldfeld,1976)

Let d be a fundamental discriminant of a real quadratic field. If there exists an elliptic curve E over $\mathbb{Q}$ such that $L(E,s)L(E^d,s)$ has a zero of order ≥ 5 at $s=1$, then for any $\epsilon > 0$ there is an effective computable constant $c_\epsilon(E) > 0$, such that

$$h(d)\log \epsilon_d > c_\epsilon(E)(\log d)^{2-\epsilon}.$$

- Goldfeld's method uses L-functions of elliptic curves.
- Without Birch-Swinnerton-Dyer conjecture for general $d > 0$ only $h(d) > (\log d)^{-\epsilon}$.

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- Without Birch-Swinnerton-Dyer conjecture for general $d > 0$ only $h(d) > (\log d)^{-\epsilon}$.
- Are there modular or automorphic forms whose L-functions have high-order zeroes at the central point (≥ 3)?

Thank you for your attention!