NUMBER THEORY SS20

HOMEWORK 1 (DUE BY 22.04.2020)

Problem 1. Define the *Dirichlet product* of the arithmetic functions f and g by

$$f * g(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

Let

$$e(n) = \begin{cases} 1 & , \text{ if } n = 1 \\ 0 & , \text{ if } n > 1 \end{cases}$$

and

$$I(n) = 1 \text{ for all } n \in \mathbb{N}.$$

Let $\mathcal{M}$ denote the set of all multiplicative arithmetic functions.

(i) Prove that for $f, g \in \mathcal{M}$ we have $f * g \in \mathcal{M}$.
(ii) Show that the Dirichlet product is associative, commutative and $e(n)$ is the identity element.
(iii) Show that every function $f \in \mathcal{M} \setminus \{f : f(1) = 0\}$ has an inverse element.

This shows that the set $\mathcal{M} \setminus \{f \equiv 0\}$ is a group under the Dirichlet product.

Problem 2. Deduce again that for any arithmetic function f we have the *Möbius inversion formula*

$$f = \mu * S_f$$

and then show that $f \in \mathcal{M}$ if and only if the sum-function $S_f \in \mathcal{M}$.

Problem 3. Show that for the number of divisors function we have the estimate

$$\tau(n) = o(n^{\varepsilon}).$$

(Hint: You can use Lemma 2 from Lecture 2.)

Problem 4. Apply the Vinogradov's lemma to show that

$$\sum_{x,y\leq N}\sum_{(x,y)=1} 1 = \frac{6}{\pi^2}N^2 + O(N\log N),$$

i.e. the density of the pairs of coprime integers up to N equals $1/\zeta(2) = 6/\pi^2$.

Problem 5 (Gauss Circle Problem)**.** Prove that for the sum of two integer squares function

$$r(n) = \sum_{x^2+y^2=n} 1$$

we have

$$\sum_{n=1}^{N} r(n) = \pi N + O(\sqrt{N}).$$

Problem 6 (Primitive Circle Problem)**.** Prove that

$$\sum_{n=1}^{N}\sum_{\substack{x^2+y^2=n\\(x,y)=1}} 1 = \frac{6}{\pi}N + O(\sqrt{N}\log N).$$