NUMBER THEORY SS20

HOMEWORK 2 (DUE BY 17.06.2020)

Problem 1. (i) Let $n \in \mathbb{N}$. Prove that

$$\int_2^x \frac{dt}{(\log t)^n} = O\left(\frac{x}{(\log x)^n}\right)$$

Hint: Split the integral into $\int_2^{f(x)} + \int_{f(x)}^x$ *for appropriate function* $f(x)$*, such that* $2 \leq f(x) < x$*, and estimate both integrals from above.*

(ii) Using integration by parts, prove that for any integer $n \geq 0$ the logarithmic integral $Li(x)$ satisfies

$$Li(x) := \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} + \frac{x}{(\log x)^2} + \ldots + \frac{n!x}{(\log x)^{n+1}} + O_n\left(\frac{x}{(\log x)^{n+2}}\right).$$

(iii) Deduce that

$$\int_2^x \frac{dt}{(\log t)^2} = \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right)$$

by first solving the integral and then applying (ii) and (i).

Problem 2 (Stirling's formula)**.** Show that for $n \to \infty$ we have the approximation

$$\log(n!) = n \log n - n + O(\log n).$$

Problem 3. Let χ be a non-principal character modulo q. Show that

$$\sum_{n \geq x} \frac{\chi(n)}{\sqrt{n}} = O\left(\frac{1}{\sqrt{x}}\right).$$

Problem 4. Given a character χ modulo q, define the *Gauss sum*

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right),$$

where $e(x) := e^{2\pi i x}$. Show that

$$\frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(a)\tau(\chi) = \begin{cases} e\left(\frac{a}{q}\right) & , (a,q) = 1; \\ 0 & , \text{otherwise.} \end{cases},$$

Problem 5 (Elementary sieves)**.** Let n denote a positive integer.

(i) (Eratosthenes-Legendre) Let P_z be the product of the primes $p \leq z$, and $\pi(x,z)$ the number of $n \leq x$ that are not divisible by any prime $p \leq z$. Show that

$$\pi(x,z) = \sum_{d|P_z} \mu(d) \left[\frac{x}{d}\right].$$

(ii) (Rankin's trick) Let $\Phi(x,z)$ be the number of $n \leq x$ all of whose prime factors are less than or equal to z. Prove that for any $\delta > 0$

$$\Phi(x,z) \leq x^\delta \prod_{p \leq z} \left(1 - \frac{1}{p^\delta}\right)^{-1}.$$

Problem 6 (*)**.** Prove that the Prime number theorem for the Chebyshev's function, i.e. the statement $\Psi(x) \sim x$, holds if and only if we have

$$\sum_{n \leq x} \mu(n) = o(x),$$

where μ is the Möbius function.

Hint: You can see Chapter 8, 'Multiplicative NT', Montgomery-Vaughan.