# Exercise 6

Chebyshev's formulation of PNT

**Theorem:**

$$M(x) := \sum_{n \leq x} \mu(n) = o(x) \iff \psi(x) \sim x.$$

**Proof, Part 1** („$\Leftarrow$") Assume $\psi(x) \sim x$.

Instead of treating $M(x)$ directly, we consider

$$N(x) := \sum_{n \leq x} \mu(n) \log n.$$

**Claim 1:** $M(x) = o(x) \iff N(x) = o(x \log x)$.

**Proof:**

$$M(x)\log x - N(x) = \sum_{n \leq x} \underbrace{\mu(n)}_{\leq 1} \log(x/n) \leq$$

$$\leq \sum_{n \leq x} \log(x/n) = \sum_{n \leq x} (\log x - \log n) \leq$$

$$\leq x \log x - \underbrace{\sum_{n \leq x} \log n}_{\log(n!)} =$$

$$\overset{\text{Stirling's formula}}{\leq} x\log x - x\log x + x + O(\log x) = O(x).$$

1

**Claim 2:**

$$\mu(n) \log n = -\sum_{d|n} \mu(d) \Lambda(n/d).$$

**Proof:**

We know that $D_{\log \cdot \mu}(s) = -D_\mu'(s)$

and so

$$D_{\log \cdot \mu}(s) = -\left(\frac{1}{\zeta(s)}\right)' = \frac{\zeta'(s)}{\zeta^2(s)} = \frac{1}{\zeta(s)} \cdot \frac{\zeta'(s)}{\zeta(s)} =$$

$$= D_\mu(s) \cdot (-D_\Lambda(s)) = -D_{\mu * \Lambda}(s)).$$

$$\Longrightarrow \log n \, \mu(n) = -\sum_{d|n} \mu(d) \Lambda(n/d).$$

2

Now we consider

$$\sum_{d|n} \mu(d)(1 - \Lambda(n/d)) = \underbrace{\sum_{d|n} \mu(d)}_{= \begin{cases} 0, & n>1 \\ 1, & n=1 \end{cases}} - \underbrace{\sum_{d|n} \mu(d)\Lambda(n/d)}_{= \begin{cases} \mu(n)\log n, & n>1 \\ 0, & n=1 \end{cases}} =$$

$$= \begin{cases} \mu(n)\log n, & n>1 \\ 1, & n=1 \end{cases}$$

summing up

$$\Rightarrow \underbrace{\sum_{n \le x} \sum_{d|n} \mu(d)(1-\Lambda(n/d))}_{} = N(x)+1 \qquad (*)$$

$$\overset{\text{change order of summation}}{=} \underbrace{\sum_{d \le x} \mu(d)\left(\left[\tfrac{x}{d}\right] - \psi(x/d)\right)}_{=: \tilde{N}(x)}$$

We estimate $\tilde{N}(x)$ instead of $N(x)$:

From the assumption, we know that

$\forall \varepsilon>0: \exists\, C = C(\varepsilon): \quad |\psi(y) - [y]| < \varepsilon y \quad \forall y \geq C$

$$\Rightarrow \quad |\psi(x/d) - [x/d]| \le \varepsilon x/d \quad \forall\, d \le x/c. \quad (**)$$

$$\Rightarrow |\tilde{N}(x)| \le \left| \sum_{d \le x/c} \mu(d)\left(\psi(x/d) - \left[\tfrac{x}{d}\right]\right) \right| +$$

$$+ \left| \sum_{x/c < d \le x} \mu(d)\left(\psi(x/d) - \left[\tfrac{x}{d}\right]\right) \right| \le$$

$$\overset{(**)}{\le} \underbrace{\sum_{d \le x/c} \frac{\varepsilon x}{d}}_{\ll\, \varepsilon x \log x} + \underbrace{\sum_{x/c < d \le x} \frac{x}{d}}_{\ll\, x \log C} \ll \varepsilon x \log x \quad \forall \varepsilon > 0$$

$$\Rightarrow \tilde{N}(x) = o(x \log x) \overset{(*)}{\Rightarrow} N(x) = o(x \log x)$$

$$\overset{\text{Claim 1}}{\Rightarrow} M(x) = o(x).$$

Part 1 □

Proof, Part 2 ("⇒") Assume $M(x) = o(x)$.

We have $\sum_{d|n} \Lambda(d) = \log n$ and, by Möbius inversion, also

$$\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d).$$

If we sum over $n \le x$, we obtain (after changing order of summation)

$$\psi(x) = \sum_{d \le x} \mu(d)\, T(x/d)$$

with

$$T(x) = \sum_{m \le x} \log m.$$

Recall $T(x) = x \log x - x + O(\log x)$ by Euler summation.

The main term here is approximately the same as in

$$D(x) = \sum_{m \le x} \tau(m) \overset{\text{Dirichlet div. problem}}{=} x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Set $f(m) := \log m - \tau(m) + 2\gamma$ and $F(x) := \sum_{m \le x} f(x)$, (***)
then $F(x) \ll \sqrt{x}$.

Moreover,

$$\sum_{r|n} \mu(r)\, \tau(n/r) = 1 \quad \forall n \in \mathbb{N}, \qquad \sum_{d|n} \mu(d) = 0 \quad \forall n > 1.$$

$$\longrightarrow \sum_{d|n} \mu(d) f(n/d) = \begin{cases} \Lambda(n) - 1, & n > 1 \\ 2\gamma - 1, & n = 1. \end{cases}$$

summing over $n \le x$ leads to

$$\sum_{d \le x} \mu(d)\, F(x/d) = \psi(x) - [x] + 2\gamma. \qquad (****)$$

If we show $\sum_{d \le x} \mu(d) F(x/d) = o(x)$, we are done.

We use Axer's theorem:

**Theorem (Axer)**

Let $a_d$ be a sequence such that

(i) $\sum_{d \le x} a_d = o(x)$ and

(ii) $\sum_{d \le x} |a_d| \ll x$.

Suppose also that $F(x)$ is a function on $[1,\infty)$ such that

(iii) $F(x)$ has bounded variation in the interval $[1, C]$ $\forall$ $C \geq 1$ and

(iv) $F(x) \ll \frac{x}{(\log x)^c}$ for some $c > 1$.

Then

$$\sum_{d \le x} a_d F(x/d) = o(x).$$

Proof: Montgomery/Vaughan: Multiplicative Number Theory I, p. 247.

We set $a_d := \mu(d)$ and $F(x)$ as in (***), then the assumption $\sum_{n \le x} \mu(x) = o(x)$ and Axer's theorem imply $\psi(x) \sim x$, because of (***).

Part 2 $\square$