1 Maximum Flow Problem

Remember the MAXIMUM FLOW PROBLEM: Given a network (G, u, s, t), where $G = (V, A)$ is a digraph, $s, t \in V$ are two distinct vertices of G and $u\colon A \to \mathbb{R}_+$ the capacity function. A function $f\colon A \to \mathbb{R}_+$ is called a flow if $f(e) \leq u(e)$ for all $e \in A$. We define

$$\mathrm{ex}_f(v) := \sum_{e \in \mathrm{in}(v)} f(e) - \sum_{e \in \mathrm{out}(v)} f(e).$$

If f fulfills $\mathrm{ex}_f(s) \leq 0$ and $\mathrm{ex}_f(v) = 0$ for all $v \in V \setminus \{s, t\}$ we call f an s-t-flow and $\mathrm{value}(f) := -\mathrm{ex}_f(s)$ the value of f. The MAXIMUM FLOW PROBLEM asks to find an s-t-flow of maximum value.

1.1 LP modelling

Model the MAXIMUM FLOW PROBLEM as an LP.

1.2 Duality

Derive the dual of the LP of subtask 1.1. What important theorem about flows can you derive via LP duality?

1.3 Integrality

Show that if u is integral, then there exists an integral maximum s-t-flow.

2 Spanning Tree Polytope

Prove that the spanning tree polytope is TDI.

3 Total Unimodality

Let $n, p, k \in \mathbb{N}$.

$$\begin{aligned}
\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} z_i &= p \\
\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i &= p \\
\sum_{i=1}^{n} z_i &\geq p - k \\
x_i + z_i &\leq 1 && i \in [n] \\
y_i + z_i &\leq 1 && i \in [n] \\
x_i, y_i, z_i &\in \{0, 1\} && i \in [n]
\end{aligned}$$

3.1 Modelling 1

Explain what the constraints above imply for feasible vectors x, y, z.

3.2 Modelling 2

Given feasible vectors x, y, z, let $X = \{i \in [n] \colon x_i = 1 \text{ or } z_i = 1\}$ and $Y = \{i \in [n] \colon y_i = 1 \text{ or } z_i = 1\}$. Rewrite the given linear constraints in terms of equivalent set operations using $X \subseteq [n]$ and $Y \subseteq [n]$ as set variables.

3.3 Integrality

Show that the constraint matrix of the following set of linear equations and inequalities is totally unimodular.