# Introduction

## Robust Optimization

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# An Uncertain World

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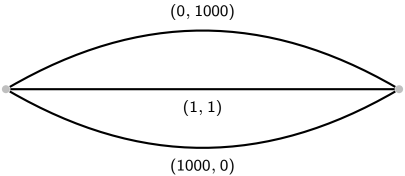

# An Uncertain World

# Scope of the course

**What is in it?**

- What is robust optimization and should we care?
- Theory (e.g. computational complexity) of RO problems.
- How to solve RO problems using a computer?
- Current research topics in RO.

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**What do you get out of it?**

- Be able to conduct your own research in RO!
- Know how to apply RO to problems in practice!

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- Mail, Discord and (virtual) office hours for questions

# Discrete Linear Optimization

**Input:**

$E = \{e_1, e_2, \ldots, e_n\}$ set of elements
$\mathcal{F} \subseteq 2^E$ set of feasible solutions
$c\colon E \to \mathbb{R}_+$ cost function

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**Binary vectors vs. sets:**
$\mathcal{X}$ set of $\{0, 1\}$ incidence vectors of feasible solutions.

# Discrete Optimization – Examples

- SELECTION $\mathcal{F} = \{X \subseteq E \colon |X| = p\}$

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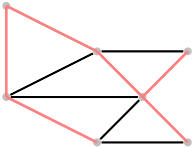
- Selection $\mathcal{F} = \{X \subseteq E \colon |X| = p\}$
- Minimum Spanning Tree



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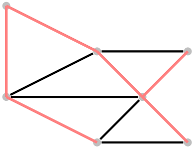
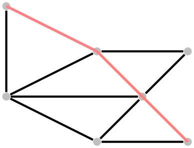


- SHORTEST PATH



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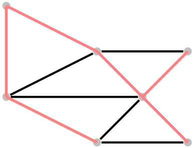

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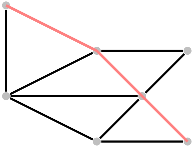

- KNAPSACK $\mathcal{F} = \{X \subseteq E \colon w(X) = \sum_{e \in X} w(e) \leq B\}$

# Robust Discrete Optimization

**Main idea:** Uncertainty sets for costs

Uncertainty set $\mathcal{U}$:
For each scenario $s$ a cost function $c^s\colon E \to \mathbb{R}_+ \in \mathcal{U}$

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**Examples**

- Discrete uncertainty $\mathcal{U}_D = \{c^{s_1}, c^{s_2}, \ldots, c^{s_K}\}$
- Interval uncertainty
  $\mathcal{U}_I = \{c^s \colon c^s(e) \in [c(e), c(e) + d(e)] \, \forall e \in E\}$

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Robust Optimization (Min-Max)

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Max-Min for maximization problems
**Also:** Uncertainty in the constraints, not only subsets/0-1 vectors, ...

# Thinking Assignment: Uncertainty Sets

- Discrete uncertainty $\mathcal{U}_D = \{c^{s_1}, c^{s_2}, \ldots, c^{s_K}\}$
- Interval uncertainty
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What other ideas do you have for modelling uncertainty using uncertainty sets?

Not a literature search exercise! Be creative!

# Example 1: Investments

Example: [Assi et al.; 2009] $\max_{x \in \mathcal{X}} \min_{p \in \mathcal{U}} \sum_{i=1}^{n} p_i x_i$

**Input:**

$b$ bound on total investment
$n$ investment opportunities
$w_i$ required cash for investment $i$
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Cash outflows and profits of the investments

| $i$ | $w_i$ | $p_i^1$ | $p_i^2$ | $p_i^3$ |
|---|---|---|---|---|
| 1 | 3 | 4 | 3 | 3 |
| 2 | 5 | 8 | 4 | 6 |
| 3 | 2 | 5 | 3 | 3 |
| 4 | 4 | 3 | 2 | 4 |
| 5 | 5 | 2 | 8 | 2 |
| 6 | 3 | 4 | 6 | 2 |

$b = 12$

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$b = 12$

**Solutions:**

Optimal values and solutions (discrete scenario case)

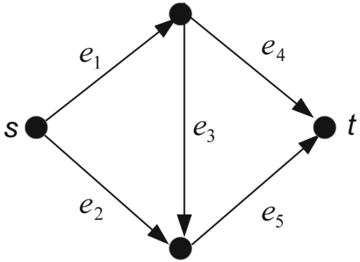
| 1 | 2 | 3 | Optimal solution |
|---|---|---|---|
| 17 | 10 | 12 | Scenario 1: (1,1,1,0,0,0) |
| 11 | 17 | 7 | Scenario 2: (0,0,1,0,1,1) |
| 16 | 9 | 13 | Scenario 3: (0,1,1,1,0,0) |
| 15 | 12 | 12 | Max–min: (0,1,0,1,0,1) |

# Example 2: Shortest Path

Example: [Kasperski, Zielinski; 2016]

$$\min_{X \in \mathcal{F}} \max_{c \in \mathcal{U}} c(X)$$

$$c^{s_1} = (2, 10, 3, 1, 1), c^{s_2} = (1, 11, 0, 5, 1), c^{s_3} = (8, 8, 0, 8, 8, 8)$$



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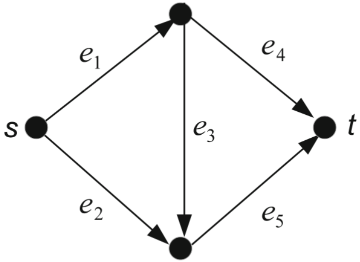


| | $c^{s_1}$ | $c^{s_2}$ | $c^{s_3}$ |
|---|---|---|---|
| $\{e_1, e_4\}$ | 3 | 6 | 16 |
| $\{e_1, e_3, e_5\}$ | 6 | 2 | 16 |
| $\{e_2, e_5\}$ | 11 | 12 | 16 |

# Thank you!