Recap: Linear Programming, (Mixed) Integer Programming

Robust Optimization

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Linear Programming

Input:
$A \in \mathbb{R}^{m \times n}$
$b \in \mathbb{R}^{m}$
$c \in \mathbb{R}^{n}$

Problem (LP):

$$\min\{c^t x \colon Ax \leq b, x \geq 0\}$$

Duality

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Dual Problem:

$$\max\{y^t b \colon y^t A \geq c, y \geq 0\}$$

Duality: Helpful overview for practical models

Primal	Dual
Equality Constraint	Free Variable
Inequality Constraint	Nonnegative Variable
Free Variable	Equality Constraint
Nonnegative Variable	Inequality Constraint

TABLE 5.1. Rules for forming the dual.

[Vanderbei]: `http://people.cs.uchicago.edu/~ivan/math/constrained_optimization_book.pdf`

(Mixed) Integer Linear Programming

IP:

$$\min\{c^t x \colon Ax \leq b, x \geq 0, x \in \mathbb{Z}^n\}$$

Binary Programming:

$$\min\{c^t x \colon Ax \leq b, x \geq 0, x \in \{0,1\}^n\}$$

Mixed Integer Programming:

$$\min\{c_I^t x + c_C^t w \colon A_I x + A_C w \leq b, x \geq 0, w \geq 0, x \in \mathbb{Z}^{n_I}\}$$

TDI and Total Unimodality

Definition

A system $Ax \leq b$ of linear inequalities is called TDI (totally dual integral) if for every integral c

$$\min\{y^t b \colon A^t y = c, y \geq 0\}$$

has an integral optimal solution y, whenever a finite minimum exists.

TDI implies integrality of the polyhedron.

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TDI implies integrality of the polyhedron.

Definition

A matrix A is totally unimodular, if all subdeterminants of A are equal to 0, $+1$ or -1.

Theorem (Hoffman, Kruskal; 1956)

A matrix A is totally uniomdular if and only if $\{x \colon Ax \leq b, x \geq 0\}$ is integral for all integral vectors b.

Total Unimodality

Theorem (Ghouila-Houri; 1962)

A matrix $A = (a_{i,j}) \in \mathbb{Z}^{m \times n}$ is totally unimodular, iff for all $R \subseteq [m]$ there exist $R_1 \dot\cup R_2 = R$ such that

$$\sum_{i \in R_1} a_{i,j} - \sum_{i \in R_2} a_{i,j} \in \{-1, 0, 1\}$$

for all $j = 1, \ldots, n$.

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Examples:

- vertex-edge incidence matrix of a bipartite graph
- vertex-arc incidence matrix of a digraph

Example: Knapsack Problem

Input:
n elements
w_i element weight
b maximum weight
p_i element profit

Problem (Knapsack):

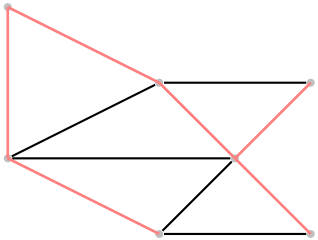
$$\max\{p^t x \colon w^t x \leq b, x \in \{0,1\}^n\}$$

Example: Spanning Tree Polytope

$$\left\{ x \in [0,1]^E : \sum_{e \in E} x_e = |V| - 1, \sum_{e \in E(G[X])} x_e \leq |X| - 1 \ \forall \emptyset \neq X \subset V \right\}$$

Theorem

The spanning tree polytope is TDI.



Separation

Definition (informal)

Given a candidate solution x a separation oracle either certifies that x is feasible, or exhibits a violated constraint.

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Theorem

There exists a polynomial time separation oracle for the spanning tree polytope.

Thank you!