Min-Max Regret Spanning Tree with Interval Uncertainties: MIP formulation

Robust Optimization

Stefan Lendl

Institute of Discrete Mathematics
Graz University of Technology

Single-Commodity Spanning Tree MIP Formulation

$$
\begin{aligned}
\min \ & \sum_{e \in E} c_e x_e \\
\text{s.t.} \ & \sum_{(i,j) \in A} f_{ij} - \sum_{(j,i) \in A} f_{ji} = \begin{cases} n-1 & \text{if } i = 1 \\ -1 & \text{if } i \in V \setminus \{1\} \end{cases} && \forall i \in V \\
& f_{ij} \leq (n-1) x_e && \forall e = \{i,j\} \in E \\
& f_{ji} \leq (n-1) x_e && \forall e = \{i,j\} \in E \\
& \sum_{e \in E} x_e = n - 1 \\
& f_{ij} \geq 0 && \forall (i,j) \in A \\
& x_e \in \mathbb{B} && \forall e \in E
\end{aligned}
$$

where $A = \{(i,j) \in V \times V \colon \{i,j\} \in E\}$.

Min-Max Regret Spanning Tree

We know

$$c^{\mathrm{wc}(T)}(e) = \begin{cases} \bar{c}(e) & \text{if } e \in T \\ \underline{c}(e) & \text{if } e \notin T \end{cases}$$

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Hence for x_e incidence vector of T it holds that

$$\max_{c \in \mathcal{U}} c(T) - c(T^*(c)) = \sum_{e \in E} \bar{c}(e) x_e - c^{\mathrm{wc}(T)}(T^*(c^{\mathrm{wc}(T)}))$$

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We would like to rewrite it into a max problem, such that we can formulate it as one big minimization problem.

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Idea: Duality
Problem: Single-Commodity Model cannot be solved as an LP.
Solution: Multi-Commodity Model!

Multi-Commodity Spanning Tree MIP Formulation

$$
\begin{aligned}
\min \quad & \sum_{\{i,j\} \in E} c_{ij}(y_{ij} = y_{ji}) \\
\text{s.t.} \quad & \sum_{(j,1) \in A} f^k_{j,1} - \sum_{(1,j) \in A} f^k_{1,j} = -1 && \forall k \in V^- \\
& \sum_{(j,i) \in A} f^k_{ji} - \sum_{(i,j) \in A} f^k_{ij} = 0 && \forall i, k \in V^-, i \neq k \\
& \sum_{(j,k) \in A} f^k_{jk} - \sum_{(k,j) \in A} f^k_{kj} = 1 && \forall k \in V^- \\
& f^k_{ij} \leq y_{ij} && \forall (i,j) \in A, k \in V^- \\
& \sum_{(i,j) \in A} y_{ij} = n - 1 \\
& f \geq 0, y \geq 0
\end{aligned}
$$

where $A = \{(i,j) \in V \times V \colon \{i,j\} \in E\}$ and $V^- = V \setminus \{1\}$.

Dual for Fixed c

$$
\begin{aligned}
\max \ & \sum_{k \in V^-} (\alpha_k^k - \alpha_1^k) + (n-1)\mu \\
\text{s.t. } & \sigma_{ij}^k \geq \alpha_j^k - \alpha_i^k && \forall (i,j) \in A, k \in V^- \\
& \sum_{k \in V^-} \sigma_{ij}^k + \mu \leq c_{ij} && \forall \{i,j\} \in E \\
& \sum_{k \in V^-} \sigma_{ji}^k + \mu \leq c_{ij} && \forall \{i,j\} \in E \\
& \sigma, \alpha \geq 0, \mu \in \mathbb{R}
\end{aligned}
$$

Modelling Worst-Case Cost

Instead of c_{ij} we write

$$
\begin{aligned}
& & & \underline{c_{ij}} + (\overline{c_{ij}} - \underline{c_{ij}}) \cdot x_{ij} \\
\text{then:} \quad & x_{ij} = 1 & \Rightarrow \quad & \underline{c_{ij}} + (\overline{c_{ij}} - \underline{c_{ij}}) \cdot 1 = \overline{c_{ij}} \\
& x_{ij} = 0 & \Rightarrow \quad & \underline{c_{ij}} + (\overline{c_{ij}} - \underline{c_{ij}}) \cdot 0 = \underline{c_{ij}}
\end{aligned}
$$

Min-Max Regret Spanning Tree MIP Formulation

$$
\begin{aligned}
\min\ & \sum_{e \in E} \bar{c}_e x_e - \sum_{k \in V^-} (\alpha_k^k - \alpha_1^k) - (n-1)\mu \\
\text{s.t. } & \sigma_{ij}^k \geq \alpha_j^k - \alpha_i^k && \forall (i,j) \in A, k \in V^- \\
& \sum_{k \in V^-} \sigma_{ij}^k + \mu \leq \underline{c}_{ij} + (\bar{c}_{ij} - \underline{c}_{ij}) x_{ij} && \forall \{i,j\} \in E \\
& \sum_{k \in V^-} \sigma_{ji}^k + \mu \leq \underline{c}_{ij} + (\bar{c}_{ij} - \underline{c}_{ij}) x_{ij} && \forall \{i,j\} \in E \\
& \sum_{(i,j) \in A} f_{ij} - \sum_{(j,i) \in A} f_{ji} = \begin{cases} n-1 & \text{if } i = 1 \\ -1 & \text{if } i \in V^- \end{cases} && \forall i \in V \\
& f_{ij} \leq (n-1) x_e && \forall e = \{i,j\} \in E \\
& f_{ji} \leq (n-1) x_e && \forall e = \{i,j\} \in E \\
& \sum_{e \in E} x_e = n-1 \\
& f, \sigma, \alpha \geq 0, \mu \in \mathbb{R} \\
& x_e \in \mathbb{B} && \forall e \in E
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Thank you!