Interval Uncertainties

Robust Optimization

Stefan Lendl

Institute of Discrete Mathematics
Graz University of Technology

Interval Uncertainties

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Theorem

The MIN-MAX *robust optimization problem for $\mathcal{X}$ with interval uncertainties $\mathcal{U}_I$ can be solved using just one execution of the classic linear optimization problem for $\mathcal{X}$.*

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Proof.

Solve an instance of the classic linear optimization problem for $\mathcal{X}$ with linear costs g given by the upper bounds $g(e) = c(e) + d(e)$. $\square$

Interval Uncertainties and Regret

Min-Max Regret

$$\min_{x \in \mathcal{X}} \max_{c \in \mathcal{U}} c(x) - \mathrm{opt}(c)$$

where

$$\mathrm{opt}(c) = \min_{x \in \mathcal{X}} c(x)$$

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Lemma

For the MIN-MAX REGRET *problem the worst case scenario* $c^{-(x)} \in \mathcal{U}_I$ *is*

$$c^{-(x)}(e) = \begin{cases} c^{+}(e) & \text{if } x_e = 1 \\ c^{-}(e) & \text{if } x_e = 0. \end{cases}$$

Lemma

For the Min-Max Regret *problem the worst case scenario* $c^{-(x)} \in \mathcal{U}_I$ *for any solution* x *is*

$$c^{-(x)}(e) = \begin{cases} c^{+}(e) & \text{if } x_e = 1 \\ c^{-}(e) & \text{if } x_e = 0. \end{cases}$$

Proof.

Let $I(x) = \{e \colon x_e = 1\}$.

$$r(x, c) = c(I(x)) - c(I(x^*(c))) = c(I(x) \setminus I(x^*(c)) - c(I(x^*(c)) \setminus I(x))$$

$$\leq c^{+}(I(x) \setminus I(x^*(c)) - c^{-}(I(x^*(c)) \setminus I(x)))$$

$$= c^{-(x)}(x) - c^{-(x)}(x^*(c))$$

$$\leq c^{-(x)}(x) - c^{-(x)}(x^*(c^{-(x)}(x)))$$

□

Properties of the optimal solution

Theorem

For the MIN-MAX REGRET *problem for* $\mathcal{X}$ *the optimal solution* x^* *corresponds to an optimal solution for its most favorable scenario* $c^{+(x^*)}$ *defined as*

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We call an interval $[a, a] = \{a\}$ degenerate.

Corollary

If the number of nondegenerate intervals in $\mathcal{U}_I$ *is* d*, the* MIN-MAX REGRET *robust optimization problem for* $\mathcal{X}$ *can be solved using* 2^d *calls to the classic linear optimization problem for* $\mathcal{X}$.

The proof is an exercise!

Approximation

Theorem

Given an instance of MIN-MAX REGRET *for* $\mathcal{X}$ *and let* x^* *be its optimal solution. Let* $c'(e) = \frac{1}{2}(c^-(e) + c^+(e))$ *and* x' *the optimal solution to the linear optimization problem for* $\mathcal{X}$ *with respect to* c'*. Then for the regret of* x' *it holds that* $r(x') \leq 2r(x^*)$*.*

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Lemma

Let $X, Y \in \mathcal{F}$*. Then*

$$r(Y) \geq c^+(Y \setminus X) - c^-(X \setminus Y) \tag{1}$$

$$r(Y) \leq r(X) + c^+(Y \setminus X) - c^-(X \setminus Y) \tag{2}$$

Computational Complexity: Interval Uncertainty

Problems	Min–max regret
SHORTEST PATH	Strongly *NP*-hard [18]
SPANNING TREE	Strongly *NP*-hard [18]
ASSIGNMENT	Strongly *NP*-hard [1]
KNAPSACK	*NP*-hard
MIN CUT	Polynomial [2]
MIN s—t CUT	Strongly *NP*-hard [2]

NP-Hardness: Spanning Tree

Theorem

The Min-Max Regret Spanning Tree *problem with interval uncertainty $\mathcal{U}_I$ is NP-hard.*

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EXACT COVER BY 3-SETS

Given: $q \in \mathbb{N}$, finite set B, $|B| = 3q$, family T of 3-element subsets of B.

Question: Does T contain an exact cover for B, i.e. $T' \subseteq T$ such that every element of B occurs in exactly one memeber of T'?

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AUXILIARY PROBLEM

Given: Connected graph G'.

Task: Find the maximum number of connected components that can be obtained from G' by removing the edges of some spanning tree in G'.

Thank you!