Budgeted Interval Uncertainties and Extensions

Robust Optimization

Stefan Lendl

Institute of Discrete Mathematics
Graz University of Technology

Budgeted Interval Uncertainties

Classic interval uncertainty:

$$\mathcal{U}_I = \{c^s \colon c^s_e \in [\hat{c}_e, \hat{c}_e + d_e] \, \forall e \in E\}$$

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(Discrete) Budgeted interval uncertainty:

$$\mathcal{U}_I^\Gamma = \{c^s \colon c_e^s \in [\hat{c}_e, \hat{c}_e + \delta_e d_e], \delta_e \in \{0,1\} \forall e \in E, \sum_{e \in E} \delta_e \leq \Gamma\}$$

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(Continous) Budgeted interval uncertainty:

$$\mathcal{U}_I^{c,\Gamma} = \{c^s \colon c^s_e \in [\hat{c}_e, \hat{c}_e + \delta_e], 0 \leq \delta_e \leq d_e \forall e \in E, \sum_{e \in E} \delta_e \leq \Gamma\}$$

Algorithm Based on Nominal Problem

Theorem

The problem MIN-MAX ROBUST OPTIMIZATION *problem with discrete budgeted uncertainty,*

$$\min_{x \in \mathcal{X}} \max_{c \in \mathcal{U}_I^\Gamma} c(x)$$

can be solved by solving $n+1$ nominal problems with respect to $\mathcal{X}$.

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Proof: Dual of adversarial problem and mathematical insights.

Algorithm Based on Nominal Problem

Theorem

The problem Min-Max Robust Optimization *problem with continous budgeted uncertainty,*

$$\min_{x \in \mathcal{X}} \max_{c \in \mathcal{U}_I^{c,\Gamma}} c(x)$$

can be solved by solving 2 *nominal problems with respect to* $\mathcal{X}$*.*

Algorithm Based on Nominal Problem

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The problem MIN-MAX ROBUST OPTIMIZATION *problem with continous budgeted uncertainty,*

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Locally Budgeted Uncertainty

$$\mathcal{U}_l^{P,\Gamma} = \{c = \hat{c} + \delta \colon 0 \le \delta_e \le d_e \forall e \in E, \sum_{i \in P_j} \delta_e \le \Gamma_j \forall j \in [K]\}$$

where $P_1 \cup \dots \cup P_K = E$ is a partition of E into regions.

What are regions?

- Periods in multi-period problems (weeks, months)
- Geographical regions
- Types of roads
- Asset classes or sectors

Overview of Computational Complexity

K number of regions

Problem	$K = O(1)$	arbitrary K
Unconstrained	P	P
Selection	P	P
Knapsack	weakly NP-hard	weakly NP-hard
Repr. Selection	P	strongly NP-hard
Spanning Tree	P	strongly NP-hard
s-t-min-cut	P	strongly NP-hard
Shortest Path	P	strongly NP-hard

Compact MIP Formulation

Adverserial Problem

$$
\begin{aligned}
\max \quad & \sum_{i \in E} (\underline{c}_i + \delta_i) x_i & \\
\text{s.t.} \quad & \sum_{i \in P_j} \delta_i \le \Gamma_j & \forall j \in [K] \\
& \delta_i \le d_i & \forall i \in E \\
& \delta_i \ge 0 & \forall i \in E
\end{aligned}
$$

Compact MIP Formulation

LP Duality:

$$\begin{aligned}
\min \ & \sum_{j\in[K]} \left(\Gamma_j \pi_j + \sum_{i\in P_j} d_i \rho_i + \sum_{i\in P_j} \underline{c}_i x_i \right) \\
\text{s.t. } & \pi_j + \rho_i \geq x_i && \forall j \in [K], i \in P_j \\
& \pi_j \geq 0 && \forall j \in [K] \\
& \rho_i \geq 0 && \forall i \in E
\end{aligned}$$

Compact MIP Formulation

Compact MIP:

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& \pi_j \geq 0 & \forall j \in [K] \\
& \rho_i \geq 0 & \forall i \in E \\
& \boldsymbol{x} \in \mathcal{X} &
\end{aligned}
$$

Constant Number of Regions

Theorem

The robust problem with locally budgeted uncertainty can be decomposed into 2^K subproblems of nominal typ, where K is the number of regions.

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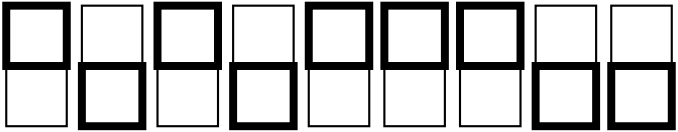
\min \ & \sum_{j \in [K]} \left(\Gamma_j \pi_j + \sum_{i \in P_j} d_i \rho_i + \sum_{i \in P_j} \underline{c}_i x_i \right) & \\
\text{s.t. } & \pi_j + \rho_i \geq x_i & \forall j \in [K], i \in P_j \\
& \pi_j \geq 0 & \forall j \in [K] \\
& \rho_i \geq 0 & \forall i \in E \\
& \boldsymbol{x} \in \mathcal{X} &
\end{aligned}
$$

Lemma

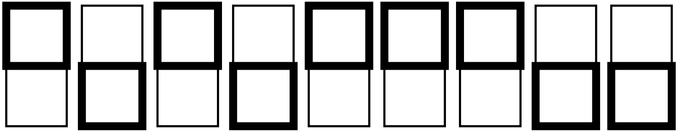
There is an optimal solution, where $\pi_j \in \{0, 1\}$ for all $j \in [K]$.

Robust Representative Selection Problem



Unbounded Number of Regions – Hardness

Robust Representative Selection Problem

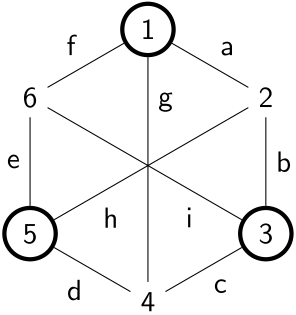


Theorem

The robust representative selection problem with locally budgeted uncertainty and arbitrary K is NP-hard, even every part contains only two candidates and for the regions it holds that $|P_j| \leq 3$ for all $j \in [K]$.

Unbounded Number of Regions – Hardness

Reduction from Vertex Cover:



Unbounded Number of Regions – Hardness

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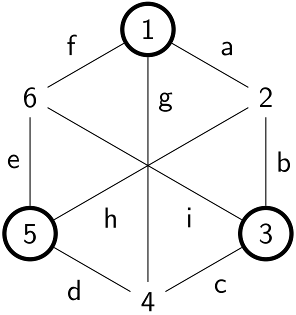

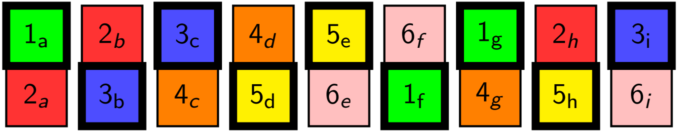

$$\underline{\boldsymbol{c}} = \mathbf{0}, \boldsymbol{d} = \mathbf{1}, \boldsymbol{\Gamma} = \mathbf{1}$$

Unbounded Number of Regions – Hardness

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The robust representative selection problem with locally budgeted uncertainty and arbitrary K is NP-hard, even every part contains only two candidates and for the regions it holds that $|P_j| \leq 3$ for all $j \in [K]$.

Theorem

Assuming ETH, there is no $O^*(2^{o(K)})$ time algorithm for the robust representative selection problem with locally budgeted uncertainty.

Unbounded Number of Regions – Selection

Theorem

The robust selection problem with locally budgeted unceratainty can be solved in $O(n \log n + pn)$ time.

Proof ideas:

- Precompute for each region j values $f_j(p_j)$: the slection problem with continous budgeted uncertainty in region j.
- Solve

$$
\begin{aligned}
T(K', p') := \min \quad & \sum_{j \in [K']} f_j(p_j) \\
\text{s.t.} \quad & \sum_{j \in [K']} p_j = p' \\
& p_j \in \mathbb{N}_0 \qquad\qquad \forall j \in [K'].
\end{aligned}
$$

using dynamic programming.

Thank you!