Pisot substitutions, numeration and tilings

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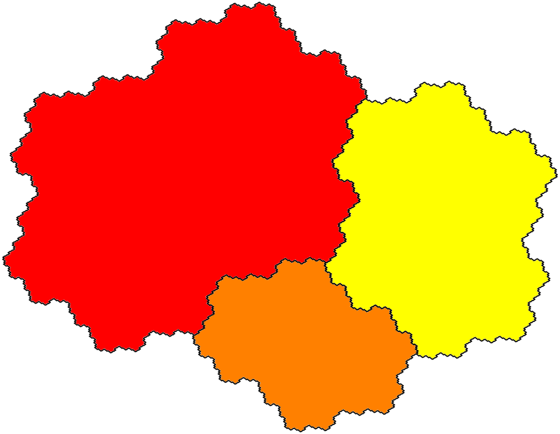

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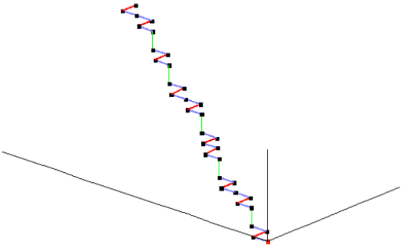

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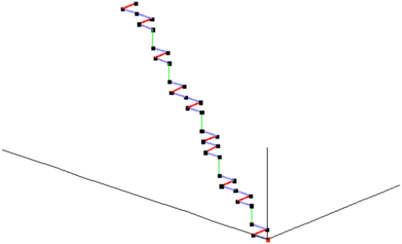

The Rauzy fractal:

$$\mathcal{R} = \overline{\{\pi \circ P(u_0 \cdots u_{n-1}) \mid n \in \mathbb{N}\}}$$

Subtiles:

$$\mathcal{R}(i) = \overline{\{\pi \circ P(u_0 \cdots u_{n-1}) \mid n \in \mathbb{N}, u_n = i\}}$$

Let $\mathcal{A}$ be a finite alphabet. A *substitution* is an endomorphism of the free monoid $\mathcal{A}^*$, i.e.,

$$\sigma : \mathcal{A}^* \to \mathcal{A}^*, \quad \sigma(uv) = \sigma(u)\sigma(v), \quad u, v \in \mathcal{A}^*.$$

We can naturally associate to a substitution σ an *incidence matrix* M_σ with entries $(M_\sigma)_{a,b} = |\sigma(b)|_a$, for all $a, b \in \mathcal{A}$.

A substitution σ is *primitive* if there exists an integer k such that, for every pair $(a, b) \in \mathcal{A}^2$, the word $\sigma^k(a)$ contains at least one occurrence of the letter b.

Definition

An algebraic integer $\alpha > 1$ is a *Pisot number* if all its algebraic conjugates α' other than α itself satisfy $|\alpha'| < 1$.

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The *prefix-suffix automaton* associated to the substitution σ is the directed graph with

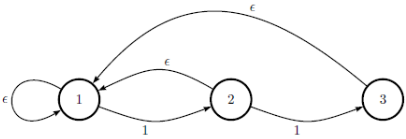
$$V = \mathcal{A}, \quad E = \{(a, b) \in \mathcal{A}^2 : \sigma(a) = pbs, \text{ for some } p, s \in \mathcal{A}^*, b \in \mathcal{A}\}.$$

We denote an edge $a \rightarrow_p b$.

- Tribonacci substitution: $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$.

$$M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \det(xI - M_\sigma) = x^3 - x^2 - x - 1.$$

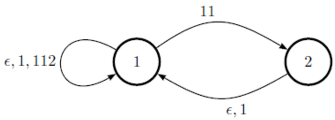
We have a real dominant eigenvalue $\beta > 1$, and two complex conjugate roots $\beta', \bar{\beta}'$ such that $|\beta'| < 1$. This is an example of *unimodular irreducible Pisot substitution*.



- Non-unit Pisot substitution: $\sigma(1) = 1121$, $\sigma(2) = 11$.

$$M_\sigma = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}, \qquad \det(xI - M_\sigma) = x^2 - 3x - 2.$$

The dominant eigenvalue is $\alpha = \frac{3+\sqrt{17}}{2}$, which is *non-unit Pisot*.



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$$\sigma^2(1) = 1213$$

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$$\sigma^3(1) = 1213121$$

Tribonacci substitution: $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$.

$$\sigma^4(1) = 1213121121312$$

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$$\sigma^5(1) = 1213121121312121312112 13$$

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$$u := \sigma^{\omega}(1) = 1213121121312121312112 13 \cdots \in \mathcal{A}^{\mathbb{N}}$$

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The main aim is to study the symbolic dynamical system (X_σ, S) generated by a primitive substitution σ:

$$X_\sigma = \overline{\{S^n u \mid n \in \mathbb{N}\}}$$

where $u \in \mathcal{A}^{\mathbb{N}}$ is a fixed point of σ and S is the shift.

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Rauzy, 1982: For the Tribonacci substitution (X_σ, S) is measure-theoretically isomorphic to a translation on the torus $\mathbb{T}^2$.

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$$T_\beta : [0,1) \longrightarrow [0,1), \qquad T_\beta(x) = \beta x - \lfloor \beta x \rfloor.$$

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Every $x \in [0,1)$ has a β-*expansion*:

$$x = \sum_{k \geq 1} d_k \beta^{-k} \qquad \longleftrightarrow \qquad (x)_\beta = .d_1 d_2 \cdots$$

where $d_k = \lfloor \beta T_\beta^{k-1}(x) \rfloor \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$.

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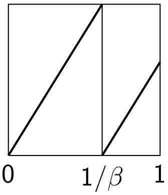
The sequences $(d_k)_{k \geq 1}$ such that $\sum_{k \geq 1} d_k \beta^{-k}$ is the β-expansion of some $x \in [0,1)$ are called *admissible*.

Set of admissible sequences forms a *subshift*: a closed shift-invariant set $X \subseteq \mathcal{A}^{\mathbb{N}}$, characterizable by a set of forbidden words.

Consider the golden ratio $\beta = \frac{1+\sqrt{5}}{2}$, which is solution of the polynomial equation $x^2 - x - 1 = 0$.

$$(1/2)_\beta = .0\overline{100}, \quad (3-\sqrt{5})_\beta = .1001, \quad (1)_\beta = .11$$



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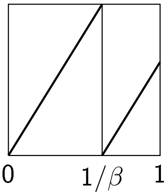

Admissibility is governed by $d^*_\beta(1) = .(10)^\omega$. Thus the subshift is equivalent to all those infinite words without any occurrence of two consecutive 1s.

- Consider the unit Pisot root β of $x^3 - x^2 - x - 1$ and let β' be one of the complex Galois conjugates of β.
- Take the set of β-integers, i.e., all those x whose β-expansion contains only non-negative powers of β:

$$x = \sum_{i=0}^{m} d_i \beta^i \in \mathbb{Z}_\beta \quad \longleftrightarrow \quad (x)_\beta = d_m d_{m-1} \cdots d_0.$$

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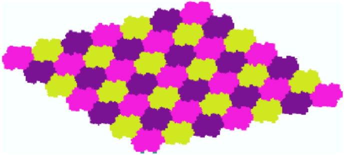
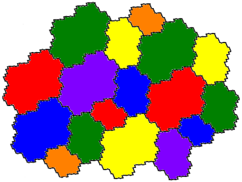
$$\mathcal{R} = \Big\{ \sum_{i \geq 0} d_i (\beta')^i : \forall i, d_i \in \{0,1\},\ d_{i+2} d_{i+1} d_i \neq 111 \Big\}.$$

- $\mathcal{R}$ is compact with non-zero measure.
- $\mathcal{R}$ is the closure of its interior and its fractal boundary has zero measure.
- $\mathcal{R}$ satisfies a *graph directed iterated function system*.
- $\mathcal{R}$ induces two different *tilings* of $\mathbb{C}$.





Consider the function $\delta : \mathcal{A}^* \to \mathbb{Q}(\alpha)$ defined by

$$\delta(p) = \langle P(p), \mathbf{v}_\alpha \rangle,$$

where $\mathbf{v}_\alpha$ is a left eigenvector of M_σ associated to α.

Theorem (Dumont, Thomas)

Let σ be a primitive substitution on the alphabet $\mathcal{A}$. Let us fix $a \in \mathcal{A}$. For every real number $x \in [0, \delta(a))$, there exists a unique (σ, a)-admissible walk $(p_i, a_i, s_i)_{i \geq 1}$ in the prefix-suffix automaton such that

$$x = \sum_{i \geq 1} \delta(p_i) \alpha^{-i}.$$

The set of digits $\mathcal{D} = \{\delta(p) \mid p \text{ prefix of } \sigma\}$ is finite and depends on $\mathbf{v}_\alpha$ and on the prefix-suffix automaton.

Set $X = \bigcup_{a\in\mathcal{A}} \big([0,\delta(a)) \times \{a\}\big)$ and consider the map

$$T_\sigma : X \to X, \quad (x,a) \mapsto \big(\alpha x - \delta(p), b\big).$$

For $x \in \mathbb{R}^+$ we have a Dumont-Thomas expansion

$$(x)_{\sigma,i} = \delta(p_n)\cdots\delta(p_0).\delta(p_{-1})\delta(p_{-2})\cdots$$

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$$\bigcup_{n\geq 0} \alpha^n \cdot T_\sigma^{-n}(0,i).$$

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- (σ, i)-integers:

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- (σ, i)-fractional parts:

$$V \cdot \mathbb{Z}[\alpha^{-1}] \cap [0, \delta(i)),$$

where $V = \langle \delta(1), \ldots, \delta(n) \rangle_{\mathbb{Z}}$.

Let K be a number field.

Ring of Integers

$$\mathcal{O}_K = \{a \in K \mid a \text{ integral over } \mathbb{Z}\} \quad \rightarrow \quad \text{Dedekind domain}$$

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$$a \in K^*, \quad a\mathcal{O}_K = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(a)}.$$

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$\mathfrak{p}$-adic valuation

Every prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ yields a $\mathfrak{p}$-*adic valuation* $v_{\mathfrak{p}}$ over K:

$$\begin{aligned} v_{\mathfrak{p}} : K^* &\longrightarrow \mathbb{Z} \\ a &\longmapsto v_{\mathfrak{p}}(a) \end{aligned}$$

Let K be a number field.

Let $K_{\mathfrak{p}}$ be the completion of K w.r.t. $|\cdot|_{\mathfrak{p}}$.

- If $\mathfrak{p} \mid \infty$, the abs. value $|\cdot|_{\mathfrak{p}}$ is defined by the Galois embeddings $\tau : K \to \mathbb{C}$. $K_{\mathfrak{p}} \cong \mathbb{R}$ or $\mathbb{C}$, depending whether $\mathfrak{p}$ is real or complex.

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- If $\mathfrak{p}$ is finite, we define

$$|a|_{\mathfrak{p}} = \mathfrak{N}(\mathfrak{p})^{-v_{\mathfrak{p}}(a)},$$

where $\mathfrak{N}(\mathfrak{p}) = p^f$ is the norm of the ideal $\mathfrak{p}$, where f is the inertia degree of $\mathfrak{p}$ lying over (p), and $v_{\mathfrak{p}}$ the $\mathfrak{p}$-adic valuation. $K_{\mathfrak{p}}$ is an extension of degree $e_{\mathfrak{p}|(p)} \cdot f_{\mathfrak{p}|(p)}$ of $\mathbb{Q}_p$.

$K_\mathfrak{p}$ completion of K w.r.t. $|\cdot|_\mathfrak{p}$:

$$\mathfrak{p} \mid \infty, \qquad K_\mathfrak{p} \cong \mathbb{R} \text{ or } \mathbb{C}$$

$$\mathfrak{p} \nmid \infty, \qquad K_\mathfrak{p} \cong \text{finite extension of } \mathbb{Q}_p$$

These are all the possible completions!

Consider the number field $K = \mathbb{Q}(\alpha)$.

The *representation space* for the substitution σ is defined as

$$K_\sigma = K'_\infty \times \prod_{\mathfrak{p} | (\alpha)} K_\mathfrak{p} = \prod_{\mathfrak{p} \in S_\alpha} K_\mathfrak{p},$$

We have the diagonal embedding

$$\begin{aligned} \Phi : \mathbb{Q}(\alpha) &\longrightarrow K_\sigma \\ \xi &\longmapsto \prod_{\mathfrak{p} \in S_\alpha} \xi \end{aligned}$$

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Lemma

Let μ be the Haar measure on K_σ. Then, for a measurable set $M \subset K_\sigma$,

$$\mu(\alpha \cdot M) = \alpha^{-1} \cdot \mu(M).$$

Set of (σ, i)-integers:

$$\mathbb{Z}_{\sigma,i} := \bigcup_{n \geq 0} \alpha^n \cdot T_\sigma^{-n}(0, i).$$

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Projecting by Φ into the representation space:

- **Subtiles** of the substitution:

$$\mathcal{T}_\sigma(i) = \overline{\Phi(\mathbb{Z}_{\sigma,i})}.$$

- **Central tile**:

$$\mathcal{T}_\sigma = \bigcup_{i \in \mathcal{A}} \mathcal{T}_\sigma(i).$$

The subtiles $\mathcal{T}_\sigma(i)$ are solutions of the following graph-directed iterated function system:

$$\forall\ i \in \mathcal{A}, \quad \mathcal{T}_\sigma(i) = \bigcup_{j \xrightarrow{p} i} \alpha \mathcal{T}_\sigma(j) + \Phi(\delta(p)).$$

Furthermore this union is measure disjoint.

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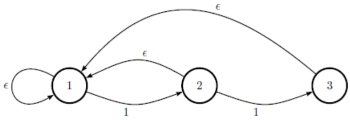
Furthermore this union is measure disjoint.

Tribonacci substitution:

$$\begin{aligned}
\mathcal{T}_\sigma(1) &= \alpha\mathcal{T}_\sigma(1) \cup \alpha\mathcal{T}_\sigma(2) \cup \alpha\mathcal{T}_\sigma(3),\\
\mathcal{T}_\sigma(2) &= \alpha\mathcal{T}_\sigma(1) + \Phi(\delta(1)),\\
\mathcal{T}_\sigma(3) &= \alpha\mathcal{T}_\sigma(2) + \Phi(\delta(1)).
\end{aligned}$$



- $\mathcal{T}_\sigma$ is compact with non-zero Haar measure.
- $\mathcal{T}_\sigma$ is the closure of its interior.
- The boundary $\partial\mathcal{T}_\sigma$ has zero Haar measure.
- The unions in the GIFS are measure-disjoint.

Consider the non-unit Pisot substitution $\sigma(1) = 1^5 2$, $\sigma(2) = 1^3$,

$$M_\sigma = \begin{pmatrix} 5 & 3 \\ 1 & 0 \end{pmatrix}, \qquad \det(xI - M_\sigma) = x^2 - 5x - 3.$$

Pisot root: $\alpha = \frac{5+\sqrt{37}}{2}$.

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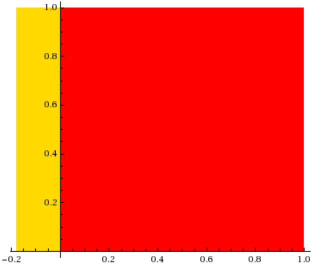
Representation space: $K_\sigma = \mathbb{R} \times \mathbb{Q}_3$.

$$\begin{aligned} \Phi : \mathbb{Q}(\alpha) &\longrightarrow \mathbb{R} \times \mathbb{Q}_3 \\ a_0 + a_1\alpha &\longmapsto (a_0 + a_1\bar{\alpha}, a_0 + a_1\alpha) \end{aligned}$$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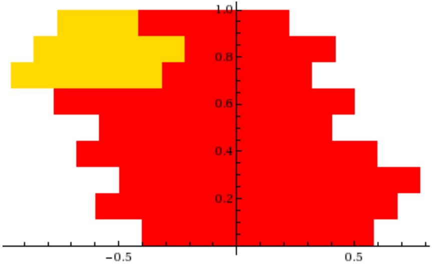
An example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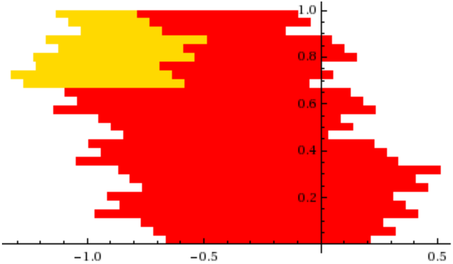
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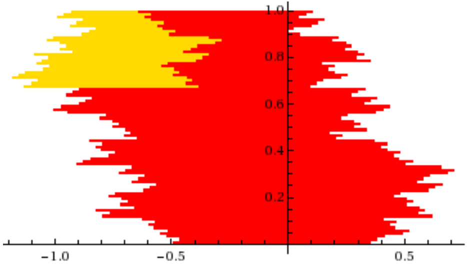
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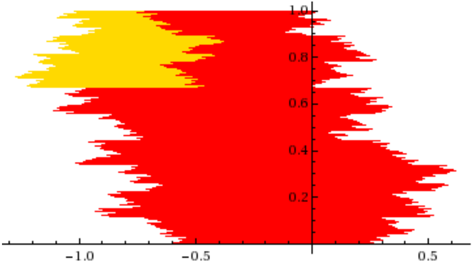
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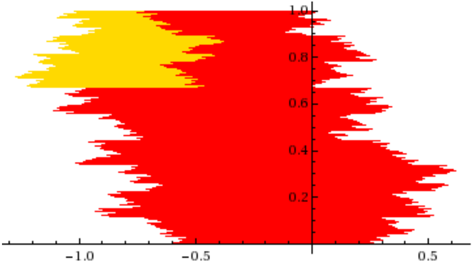

$$\mathcal{T}_\sigma \subset \mathbb{R} \times \mathbb{Z}_3$$

$$\mathcal{T}_\sigma(2) = \mathcal{T}_{\delta(1^5).}, \quad \mathcal{T}_\sigma(1) = \mathcal{T}_\sigma \setminus \mathcal{T}_\sigma(2).$$

Translation set:

$$\Gamma = \{(\Phi(x), i) \in K_\sigma \times \mathcal{A} \mid x \in V \cdot \mathbb{Z}[\alpha^{-1}] \cap [0, \delta(i))\}$$

Γ is a *Delone set*, i.e., uniformly discrete and relatively dense.

Theorem (Thuswaldner, M.)

The collection $\{\mathcal{T}_\sigma(i) + \gamma \mid (\gamma, i) \in \Gamma\}$ forms a *self-replicating multiple tiling* of K_σ.

Two important concepts:

- Exclusivity of a point of a tile.
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Lemma

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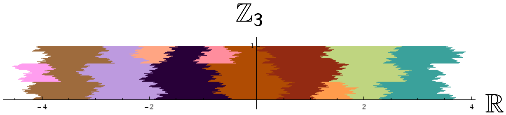
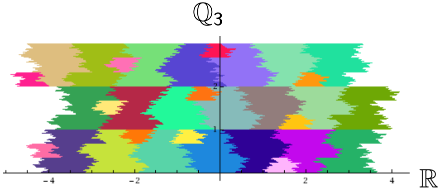
The substitution σ satisfies the geometric property (F) if and only if 0 is an exclusive inner point of the central tile $\mathcal{T}_\sigma$.

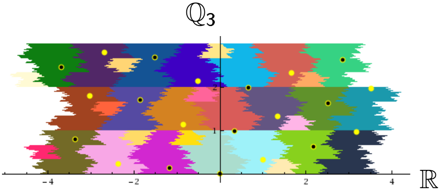
Theorem (Thuswaldner, M.)

Let σ be an irreducible Pisot substitution. If σ satisfies the geometric property (F), the self-replicating multiple tiling $\{\mathcal{T}_\sigma(i) + \gamma \mid (\gamma, i) \in \Gamma\}$ is a tiling.


$\mathbb{Z}_3$
$\mathbb{R}$
-4
-2
2
4


$\mathbb{Q}_3$
$\mathbb{R}$
-4
-2
2
4

Tiling associated to $\sigma(1) = 1^5 2$, $\sigma(2) = 1^3$, with translation set.



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Pisot conjecture

- Every unimodular irreducible Pisot substitution σ induces a lattice tiling and a self-replicating tiling of its representation space.
- (X_σ, S) has pure discrete spectrum or, equivalently, is measure-theoretically isomorphic to a translation on the torus $\mathbb{T}^{n-1}$.

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- We obtain a multiple tiling and tiling conditions.

Thanks for the attention!