Tilings for Pisot beta-numeration

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Overview

1. Fractals and numeration: an instructive example.
2. Beta-numeration and Rauzy fractals.
3. Tilings.

Knuth's numeration system

Every element of the ring of integers $\mathbb{Z}[i]$ of the number field $\mathbb{Q}(i)$ admits a unique representation

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- $(-1+i, \{0,1\})$ is a CNS for $\mathbb{Z}[i]$.
- Basis $b = -n + i$, $n \geq 1$, and digits $\{0, 1, \ldots, |N(b)| - 1\}$ give a CNS for $\mathbb{Z}[i]$.
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Consider the set of "fractional parts"

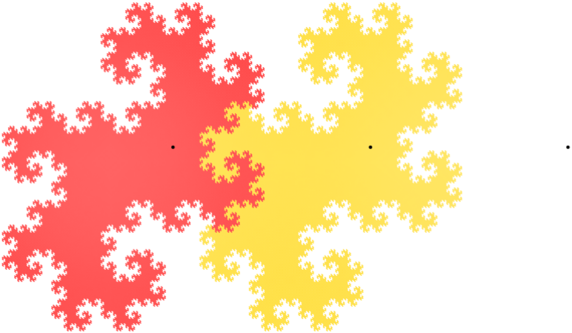
$$\mathcal{T} = \left\{ z \in \mathbb{C} \mid z = \sum_{k\geq 1} d_k(-1+i)^{-k},\ d_k \in \{0,1\} \right\}.$$

Twin Dragon

- $\mathcal{T} = b^{-1}\mathcal{T} \cup b^{-1}(\mathcal{T}+1)$.
- $\mathcal{T}$ is compact with non-zero Lebesgue measure.
- $\mathcal{T}$ is the closure of the interior.
- $\partial\mathcal{T}$ has zero measure with $\dim_H \partial\mathcal{T} = 1.5236\cdots$.
- And...

Tilings

Tilings



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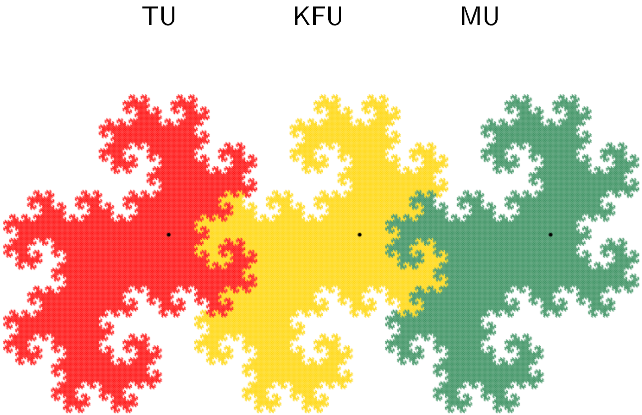

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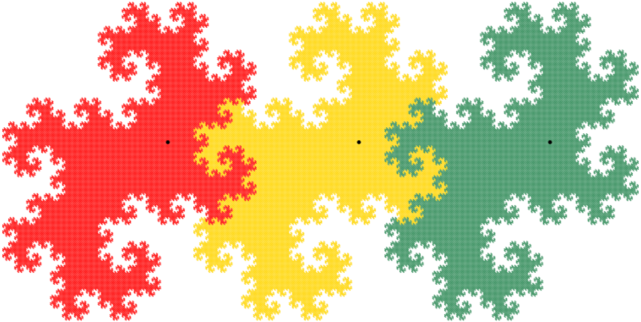

$\mathcal{T} + \mathbb{Z}[i]$ is a periodic tiling of $\mathbb{C}$

Beta numeration

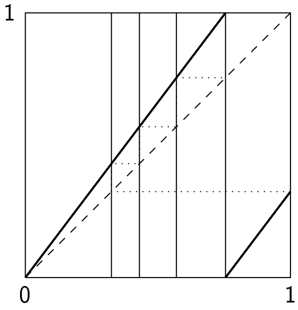

Figure : T_β for $\beta^3 = \beta + 1$.

Let $\beta > 1$ be a Pisot number. Define

$$\begin{aligned} T_\beta : [0,1) &\to [0,1) \\ x &\mapsto \beta x - \lfloor \beta x \rfloor \end{aligned}$$

Every $x \in [0,1]$ has a *(greedy) β-expansion*:

$$(x)_\beta = .d_1 d_2 d_3 \cdots$$

with $d_i \in \mathcal{A} = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$.

$([0,1), T_\beta)$ is conjugate to a either sofic or of finite type *subshift*, the admissibility depending on $(1)_\beta$.

Beta substitutions

Example: β smallest Pisot number, $(1)_\beta = .10001$

$$\sigma_\beta : 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 5,\ 5 \mapsto 1$$

$$M_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad f(X)g(X) = (X^3 - X - 1)(X^2 - X + 1)$$

σ_β *reducible unit* Pisot substitution.

Geometric interpretation

- Set of β-integers with fractional part x: $\beta^k T_\beta^{-k}(x)$.

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Rauzy fractals

$$\mathcal{R}(x) = \overline{\bigcup_{k \geq 0} \delta_c(\beta^k T_\beta^{-k}(x))} \in \mathbb{K}_\beta^c$$

Representation space

Let $K = \mathbb{Q}(\beta)$. **Representation space**:

$$\mathbb{K}_\beta := K_\infty \times \prod_{\mathfrak{p} | (\beta)} K_{\mathfrak{p}}$$

where

- $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$, abs. values given by Galois embeddings.
- $K_{\mathfrak{p}}$ finite extension of $\mathbb{Q}_p$, for $\mathfrak{p} \mid (p)$, $|\cdot|_{\mathfrak{p}} = \mathfrak{N}(\mathfrak{p})^{-v_{\mathfrak{p}}(\cdot)}$.

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Hyperbolic decomposition: $\mathbb{K}_\beta = \mathbb{K}_\beta^e \oplus \mathbb{K}_\beta^c$.
$\mathbb{K}_\beta^e = K_{\mathfrak{p}_1}$ and multiplication by β is a contraction in $\mathbb{K}_\beta^c$.

Embed K into $\mathbb{K}_\beta$, $\mathbb{K}_\beta^c$ diagonally by δ, δ_c.

How they look like

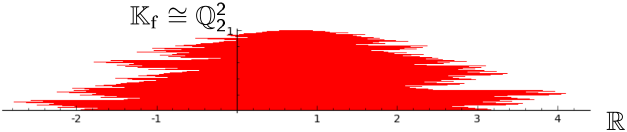
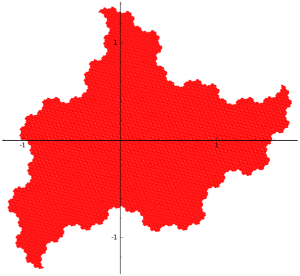


Figure : $\mathcal{R}(0)$ for $\beta^2 = 2\beta + 2$.



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Properties of Rauzy fractals

Rauzy fractals

- are compact with non-zero Haar measure.
- are the closure of their interior.
- have fractal boundary with zero Haar measure.
- are self-similar (IFS)

$$\mathcal{R}(x) = \bigcup_{y \in T_\beta^{-1}(x)} \beta \mathcal{R}(y)$$

Self-similar structure

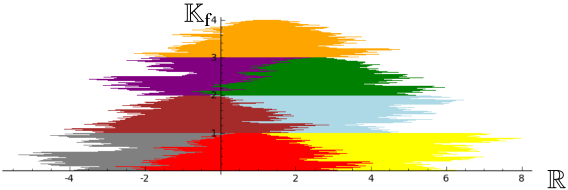


Figure : $\beta^{-2}\mathcal{R}(0)$ for $\beta^2 = 2\beta + 2$.

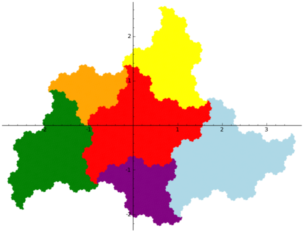


Figure : $\beta^{-5}\mathcal{R}(0)$ for $\beta^3 = \beta + 1$.

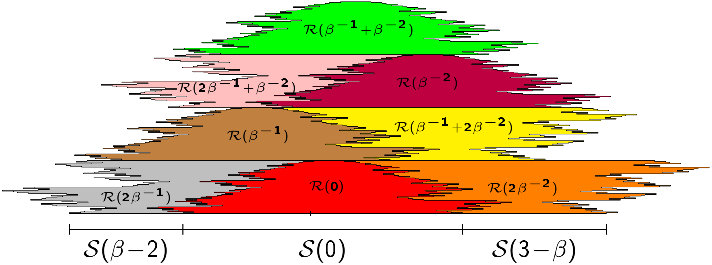
Integral beta-tiles

Framework: β **non-unit**.

Integral β-tiles

For $x \in \mathbb{Z}[\beta] \cap [0,1)$,

$$\mathcal{S}(x) = \{(z_\mathfrak{p}) \in \mathcal{R}(x) : z_\mathfrak{p} = 0 \ \text{ for each } \ \mathfrak{p} \mid (\beta)\}$$



Properties of integral beta-tiles

1. $\mathcal{S}(x)$ form "slices" of $\mathcal{R}(0)$.
   (Main tool: Strong approximation theorem)
2. $\mathcal{S}(x) \neq \emptyset$ iff $x \in \mathbb{Z}[\beta]$.
3. $\mathcal{S}(x) = \mathrm{Lim}_{k\to\infty}\, \delta_\infty^c(\beta^k(T_\beta^{-k}(x) \cap \mathbb{Z}[\beta])) \in K_\infty^c$.
4. $\mathcal{S}(x) - \delta_\infty^c(x)$ is close to $\mathcal{S}(y) - \delta_\infty^c(y)$ if $|x-y|_\mathfrak{p}$ is small $\forall\, \mathfrak{p} \mid (\beta)$.
5. $\mathcal{S}(x)$ are SRS tiles [Berthé, Siegel, Steiner et al. 2011].

Natural extensions

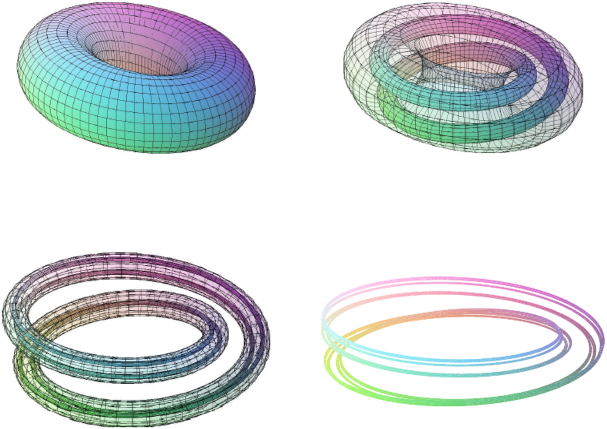


Figure : The dyadic solenoid.

Natural extensions

Suspend the Rauzy fractals with intervals...

$$\mathscr{X} = \bigcup_{i=1}^{m-1} [v_i, v_{i+1}) \times (\delta_c(v_i) - \mathcal{R}(v_i)),$$

$$\mathscr{T}_\beta : \mathscr{X} \to \mathscr{X}, \quad (x, \mathbf{y}) \mapsto \big(T_\beta(x), \beta \cdot \mathbf{y} - \delta_c(\lfloor \beta x \rfloor)\big)$$

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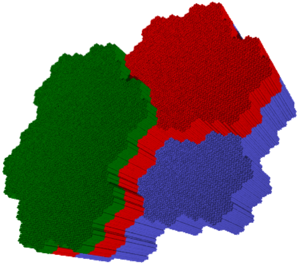
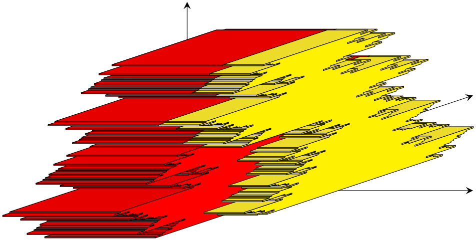
- $(\mathscr{X}, \mathscr{B}, \mu, \mathscr{T}_\beta)$ is a natural extension of $([0,1), B, \mu \circ \pi^{-1}, T_\beta)$.
- $\mathbb{K}_\beta = \mathscr{X} + \delta(\mathbb{Z}[\beta^{-1}])$.
- $x \in \mathrm{Pur}(\beta)$ if and only if $x \in \mathbb{Q}(\beta)$, $\delta(x) \in \mathscr{X}$.

Natural extensions

Tribonacci natural extension in $\mathbb{R} \times \mathbb{C}$:



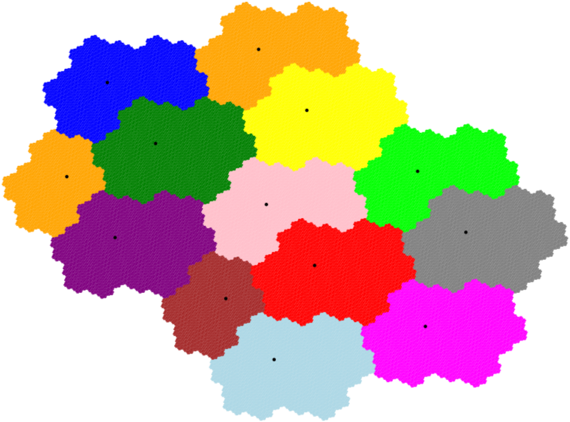
Natural extension associated to $\beta^2 = 2\beta + 2$ in $\mathbb{R}^2 \times \mathbb{Q}_2^2$:



Aperiodic tiling

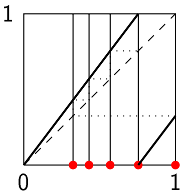
Delone set: $\delta_c(\mathbb{Z}[\beta^{-1}] \cap [0,1))$.

$\mathcal{C}_{\mathrm{aper}} = \{\mathcal{R}(x) : x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)\}$ multiple tiling of $\mathbb{K}_\beta^c$.



Periodic tiling

- $V = \{T_\beta^k(1) : k \geq 0\} \setminus \{0\}$.



Periodic tiling

- $V = \{T_\beta^k(1) : k \geq 0\} \setminus \{0\}$.
- Translation set: $\Lambda = \delta_c(\langle V - V \rangle_{\mathbb{Z}})$.

  Related to a domain exchange defined on $\mathcal{R}(0)$ conjugate to the β-substitution dynamical system (X_σ, S).

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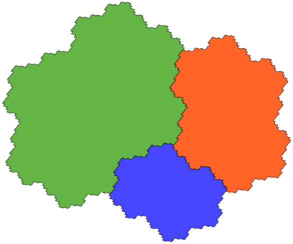

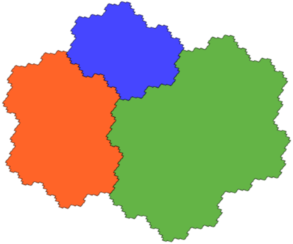

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- Problem: if $\# V > \deg(\beta)$, Λ is not always a lattice.
  Condition (QM): Λ is a lattice.
- (Non-unit β) Let $Z = K_\infty^c \times \prod_{\mathfrak{p}|(\beta)} \overline{\delta_{\mathfrak{f}}(\mathbb{Z}[\beta])}$ be the *stripe space*.

Periodic tiling

If (QM) holds, $\Lambda = \delta_c(\langle V - V \rangle_{\mathbb{Z}})$ lattice.

$\mathcal{C}_{\mathrm{per}} = \mathcal{R}(0) + \Lambda$ multiple tiling of Z.

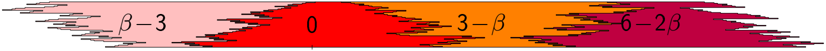


Figure : A patch of $\mathcal{C}_{\mathrm{per}}$, $\beta^2 = 2\beta + 2$.

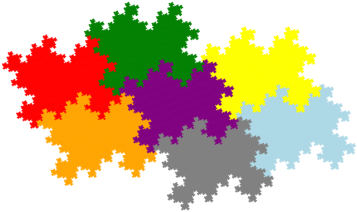
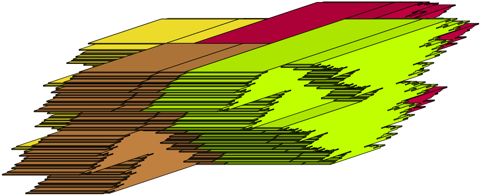


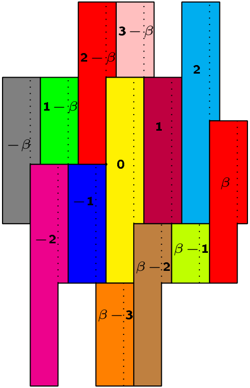
Figure : A patch of $\mathcal{C}_{\mathrm{per}}$, $\beta^3 = 2\beta^2 - \beta + 1$.

Natural extension tiling

Translation set: lattice $\delta(\mathbb{Z}[\beta^{-1}])$.

$\mathcal{C}_{\text{ext}} = \mathscr{X} + \delta(\mathbb{Z}[\beta^{-1}])$ multiple tiling of $\mathbb{K}_\beta$.

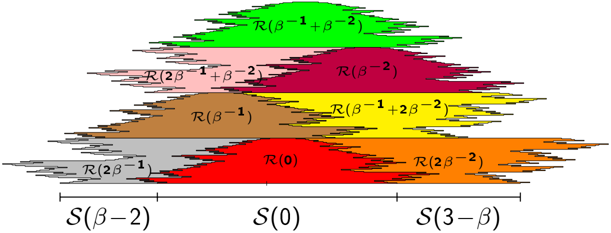




Integral beta-tiles tiling

Translation set: $\delta_\infty^c(\mathbb{Z}[\beta] \cap [0,1))$.

$\mathcal{C}_{\text{int}} = \{\mathcal{S}(x) : x \in \mathbb{Z}[\beta] \cap [0,1)\}$ weak multiple tiling of $\mathbb{K}_\infty^c$.



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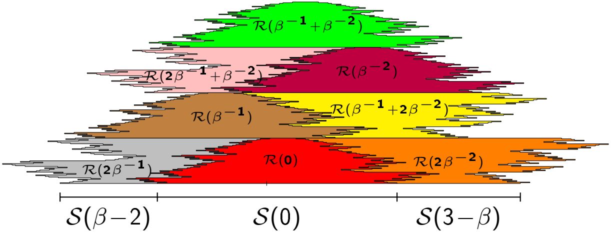

For every quadratic Pisot β, the integral beta-tiles form a tiling by intervals!

Tiling equivalences

Theorem (M., Steiner 2013)

Let β be a Pisot number. Then the collections $\mathcal{C}_{\mathrm{ext}}$, $\mathcal{C}_{\mathrm{aper}}$, and $\mathcal{C}_{\mathrm{int}}$ are multiple tilings of $\mathbb{K}_\beta$, $\mathbb{K}_\beta^c$, and $\mathbb{K}_\infty^c$, respectively, and they all have the same covering degree. The following statements are equivalent:

1. All collections $\mathcal{C}_{\mathrm{ext}}$, $\mathcal{C}_{\mathrm{aper}}$, and $\mathcal{C}_{\mathrm{int}}$ are tilings.
2. One of the collections $\mathcal{C}_{\mathrm{ext}}$, $\mathcal{C}_{\mathrm{aper}}$, and $\mathcal{C}_{\mathrm{int}}$ is a tiling.
3. One of the collections $\mathcal{C}_{\mathrm{ext}}$, $\mathcal{C}_{\mathrm{aper}}$, and $\mathcal{C}_{\mathrm{int}}$ has an exclusive point.
4. Property (W) holds.
5. The spectral radius of the boundary graph is less than β.

If (QM) holds, then the following is also equivalent to the ones above:

6. $\mathcal{C}_{\mathrm{per}}$ is a tiling of Z.

Pisot conjecture for β-numeration

Any of the above collections is a tiling.

Appendix

Property (W)

$$\forall\, x \in \mathrm{Pur}(\beta) \cap \mathbb{Z}[\beta]\ \exists\, y \in [0, 1-x),\ n \in \mathbb{N} :\ T^n(x+y) = T^n(y) = 0$$