Fractals arising from numeration and substitutions

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1. The geometrical approach for the study of substitution dynamical systems.

2. Escaping some of the hypothesis: main differences and generalizations.

Tribonacci substitution: $\mathcal{A} = \{1, 2, 3\}$, $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$.

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$$\sigma^5(1) = 1213121121312121312112 13$$

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We study the symbolic dynamical system (X_σ, S) generated by a primitive substitution σ:

$$X_\sigma = \overline{\{S^n \mathrm{u} \mid n \in \mathbb{N}\}}$$

where $\mathrm{u} \in \mathcal{A}^{\mathbb{N}}$ is a fixed point of σ and S is the shift.

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Aim: Understand the ergodic behaviour of these systems.

Rauzy, 1982: For the Tribonacci substitution (X_σ, S) is conjugate to a minimal toral translation $(\mathbb{T}^2, \tau)$.

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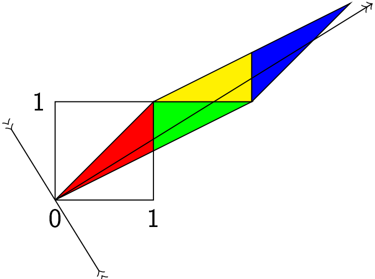
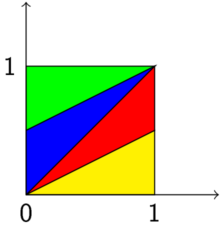
Dynamically: expanding direction $\oplus$ contracting hyperplane.

E.g. consider the hyperbolic toral automorphism $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$





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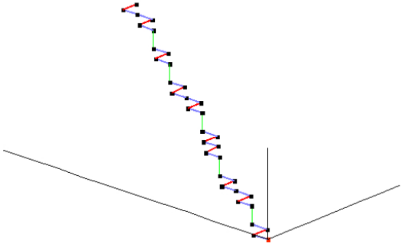

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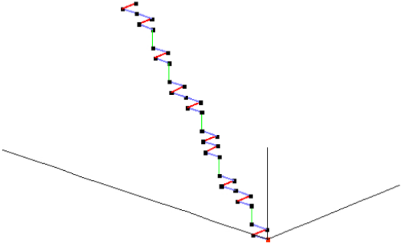
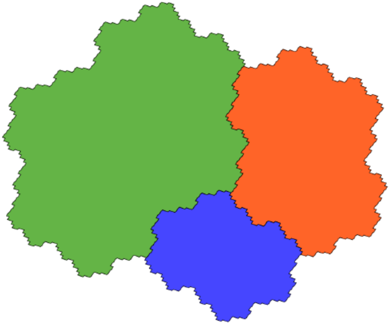
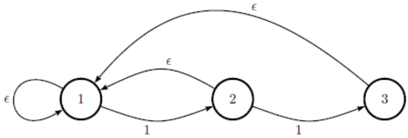




Rauzy fractal

$$\mathcal{R}_a = \overline{\{\pi_c \circ P(u_0 \cdots u_{n-1}) \mid n \in \mathbb{N}, u_n = a\}}, \quad \mathcal{R} = \bigcup_{a \in \mathcal{A}} \mathcal{R}_a.$$

- Dumont-Thomas '89: Every finite prefix of u can be uniquely expanded as $\sigma^n(d_n)\sigma^{n-1}(d_{n-1})\cdots d_0$, $d_i \in \{0,1\}$.



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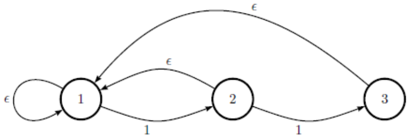

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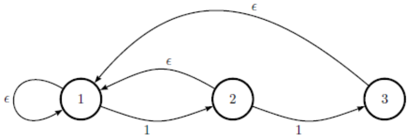

- $P \circ \sigma = M_\sigma \circ P$.
- Take the scalar product with a left eigenvector $\mathbf{v}_\beta$. Action of M_σ on $\mathbb{R}^3$ is equivalent to action of β on $\mathbb{R} \times \mathbb{C} = \mathbb{K}_e \times \mathbb{K}'_\beta$. Projection π_c $\leftrightarrow$ Embedding δ'.

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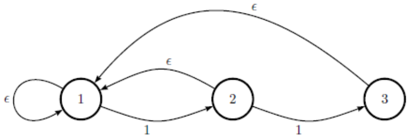

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- The points of the broken line are β-*integers*:

$$\sum_{i=0}^{n} d_i \beta^i \in \mathbb{N}_\beta$$

and projecting them and taking the closure we get the Rauzy fractal.

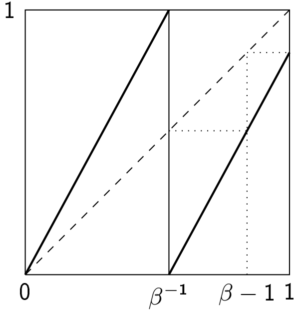


Figure : T_β for $\beta^3 = \beta^2 + \beta + 1$.

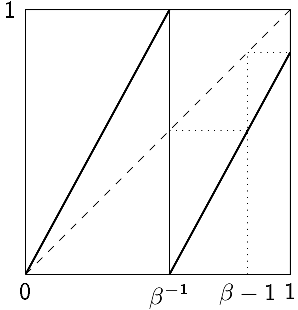
Beta-transformation:

$$\begin{aligned} T_\beta : [0,1) &\to [0,1) \\ x &\mapsto \beta x - \lfloor \beta x \rfloor \end{aligned}$$

Every real $x \in [0,1)$ has a β-*expansion*:

$$(x)_\beta = .d_1 d_2 d_3 \cdots$$

with $d_i \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$. Not every word is allowed! This depends on $(1)_\beta$.



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$$\begin{aligned} V &= \{T_\beta^k(1) : k \geq 0\} \\ &= \{1, \beta - 1, \beta^{-1}, 0\} \end{aligned}$$

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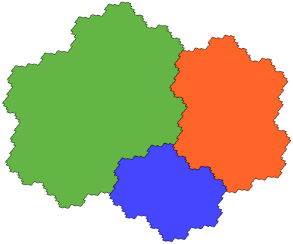
$$x + \sum_{i=0}^{n} d_i \beta^i \in \bigcup_{k \geq 0} \beta^k T_\beta^{-k}(x)$$

and embedding we get a translated Rauzy fractal

$$\mathcal{R}(x) = \overline{\bigcup_{k \geq 0} \delta'(\beta^k T_\beta^{-k}(x))}$$

How does the action of S on X_σ translate on $\mathcal{R}$?

Recall: $\mathcal{R}_a = \overline{\{\pi_c \circ P(u_0 \cdots u_{n-1}) \mid n \in \mathbb{N}, u_n = a\}}$ and Dumont-Thomas



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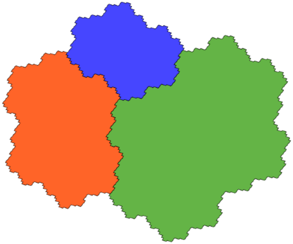

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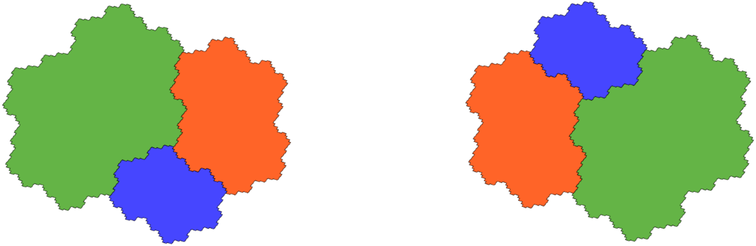

The Strong Coincidence Condition (SCC) holds: the subtiles are disjoint in measure.

Domain exchange transformation:

$$E : \mathcal{R} \to \mathcal{R}, \quad \mathbf{z} \mapsto \mathbf{z} + \delta'(v_a) \quad \text{for} \quad \mathbf{z} \in \mathcal{R}_a.$$

The action of E on $\mathcal{R}$ is coded by (X_σ, S) thanks to the partition given by the subtiles.

$$\begin{tikzcd} X_\sigma \arrow[r] \arrow[d, "S"'] & \mathcal{R} \arrow[d, "E"'] \\ X_\sigma \arrow[r] & \mathcal{R} \end{tikzcd}$$

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$$\begin{tikzcd}
X_\sigma \arrow[r] \arrow[d, "S"'] & \mathcal{R} \arrow[r] \arrow[d, "E"'] & \mathbb{C}/\Lambda \arrow[d, "\tau"'] \\
X_\sigma \arrow[r] & \mathcal{R} \arrow[r] & \mathbb{C}/\Lambda
\end{tikzcd}$$

Now we want the conjugation with a toral translation!

Anti-diagonal lattice: $\Lambda = \delta'(\langle V - V\rangle_{\mathbb{Z}})$.
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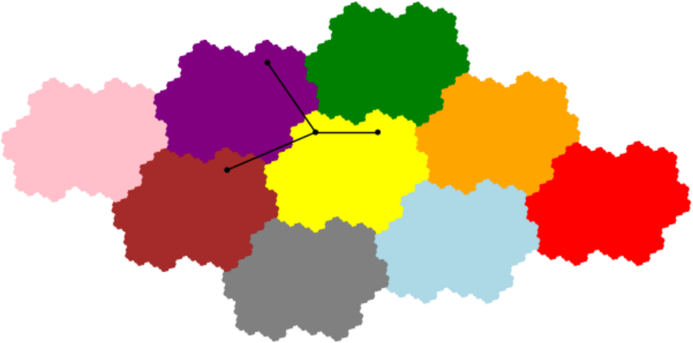

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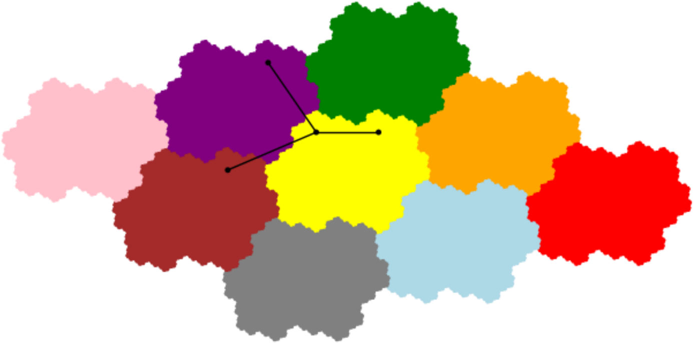


If $\mathcal{R} + \Lambda$ is a tiling of $\mathbb{C}$, then we have the conjugation with the toral translation $(\mathbb{C}/\Lambda, \tau)$.

This geometrical construction can be applied to every *irreducible unit Pisot* substitution (Arnoux, Ito 2001).

Pisot conjecture

Let σ be an irreducible unit Pisot substitution. Then (X_σ, S) has pure discrete spectrum.

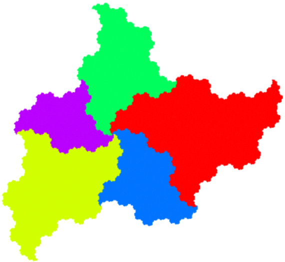
There are several combinatorial, topological and arithmetical conditions that imply the tiling property.

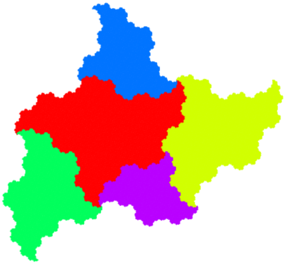
Key-words: irreducible, unit.

Example: β smallest Pisot number, $(1)_\beta = .10001$

$$\sigma_\beta : 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 5,\ 5 \mapsto 1$$

$$M_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad f(X)g(X) = (X^3 - X - 1)(X^2 - X + 1)$$





Escaping irreducibility

- Pisot conjecture is **false** for reducible Pisot substitutions (Barge, Baker, Kwapisz 2006)! Easy example: Thue-Morse.
- Remarkable fact: no example of a β-substitution failing the Pisot conjecture is known.
- Problems with periodic tiling due to $\#V > \deg(\beta)$. For β minimal Pisot $L = \langle V - V \rangle_{\mathbb{Z}} = \mathbb{Z}[\beta]$.

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$$\text{(QM):} \quad \operatorname{rank}(L) = \deg(\beta) - 1 \quad \Rightarrow \quad \delta'(L) \;\text{ lattice}$$

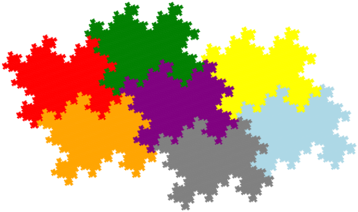

Figure : A patch of $\mathcal{C}_{\mathrm{per}}$ for $\beta^3 = 2\beta^2 - \beta + 1$.

Idea: if β is not a unit we enlarge the representation space so that β becomes a unit therein.

Let $K = \mathbb{Q}(\beta)$. **Representation space**:

$$\mathbb{K}_\beta := K_\infty \times \prod_{\mathfrak{p}|(\beta)} K_\mathfrak{p}$$

where

- $K_\infty = K \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$, abs. values given by Galois embeddings.
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$\mathbb{K}_\beta = \mathbb{K}_{\mathfrak{p}_1} \times \mathbb{K}'_\beta$. Multiplication by β is a contraction on $\mathbb{K}'_\beta$.
Embed K into $\mathbb{K}_\beta$, $\mathbb{K}'_\beta$, $\mathbb{K}'_\infty$ diagonally by δ, δ', δ'_∞.

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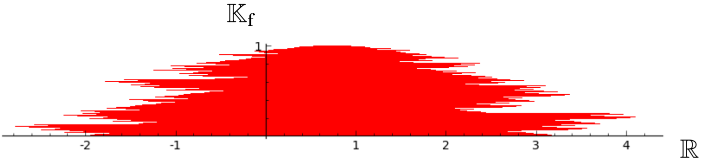
- provide an aperiodic multiple tiling of $\mathbb{K}'_\beta$:

$$\mathcal{C}_{\mathrm{aper}} = \{\mathcal{R}(x) : x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)\}$$

Non-unit example: $\beta^2 = 2\beta + 2$, $\mathbb{K}'_\beta = \mathbb{R} \times \mathbb{K}_\mathrm{f} \cong \mathbb{R} \times \mathbb{Q}_2^2$.



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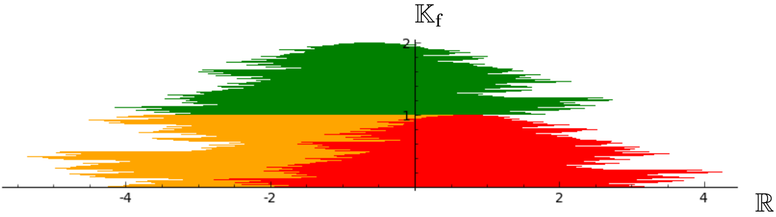

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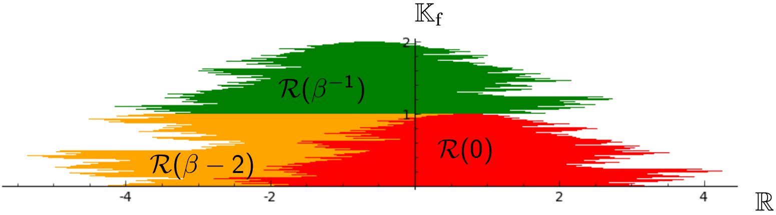

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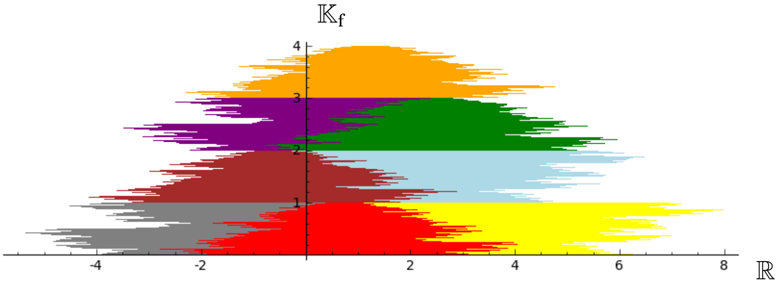

Figure : $\beta^{-2}\mathcal{R}(0)$ for $\beta^2 = 2\beta + 2$.

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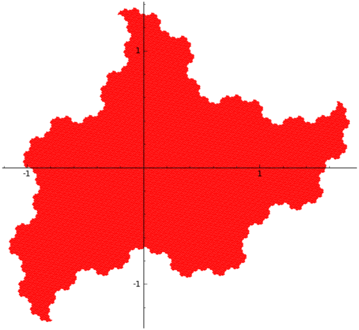

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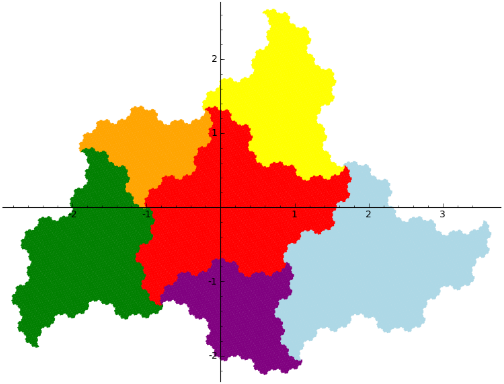

Figure : $\beta^{-5}\mathcal{R}(0)$ for $\beta^3 = \beta + 1$.

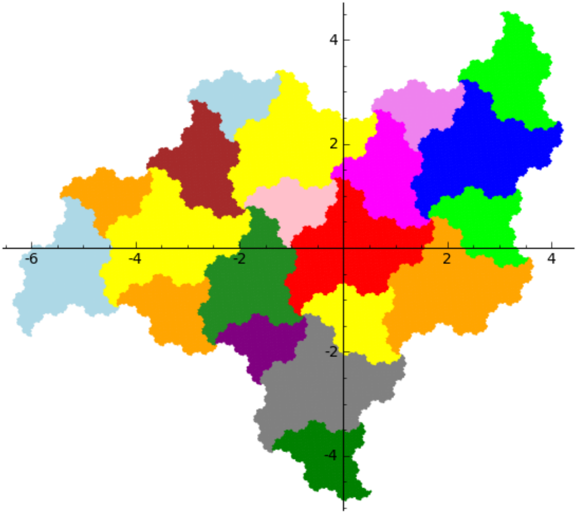

Figure : Patch of $\mathcal{C}_{\mathrm{aper}}$ for $\beta^3 = \beta + 1$.

Suspend the Rauzy fractals with intervals...

$$\mathscr{X} = \bigcup_{i=1}^{m-1} [v_i, v_{i+1}) \times (\delta'(v_i) - \mathcal{R}(v_i)),$$

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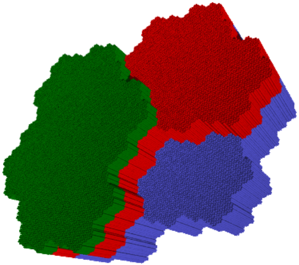
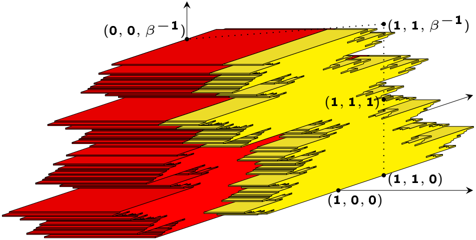
$$\mathscr{X} = \bigcup_{i=1}^{m-1} [v_i, v_{i+1}) \times (\delta'(v_i) - \mathcal{R}(v_i)),$$
$$\mathscr{T}_\beta : \mathscr{X} \to \mathscr{X}, \quad (x, \mathbf{y}) \mapsto \big(T_\beta(x), \beta \cdot \mathbf{y} - \delta'(\lfloor \beta x \rfloor)\big)$$

- $(\mathscr{X}, \mathscr{B}, \mu, \mathscr{T}_\beta)$ is a natural extension of $([0,1), B, \mu \circ \pi^{-1}, T_\beta)$.
- $\mathbb{K}_\beta = \mathscr{X} + \delta(\mathbb{Z}[\beta^{-1}])$.
- $x \in \mathrm{Pur}(\beta)$ if and only if $x \in \mathbb{Q}(\beta)$, $\delta(x) \in \mathscr{X}$.

Tribonacci natural extension in $\mathbb{R} \times \mathbb{C}$:



Natural extension associated to $\beta^2 = 2\beta + 2$ in $\mathbb{R}^2 \times \mathbb{Q}_2^2$:



What we want is that these natural extensions are conjugate to toral/solenoidal automorphisms!

Framework: β **non-unit**.

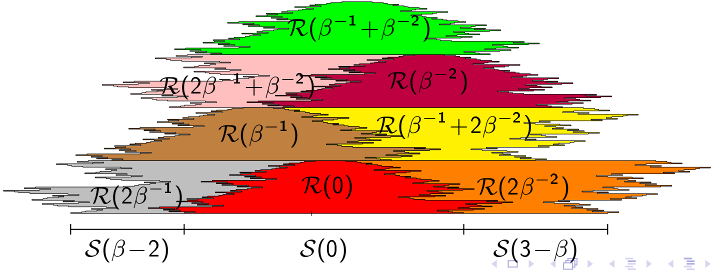
Integral β-tiles

For $x \in \mathbb{Z}[\beta] \cap [0,1)$,

$$\mathcal{S}(x) = \{(z_\mathfrak{p}) \in \mathcal{R}(x) : z_\mathfrak{p} = 0 \text{ for each } \mathfrak{p} \mid (\beta)\}$$

Properties:

1. $\mathcal{S}(x)$ form "slices" of $\mathcal{R}(0)$ and of $\mathcal{X}$.
2. $\mathcal{S}(x) \neq \emptyset$ iff $x \in \mathbb{Z}[\beta]$.
3. $\mathcal{S}(x) = \mathrm{Lim}_{k\to\infty}\, \delta'_\infty(\beta^k(T_\beta^{-k}(x) \cap \mathbb{Z}[\beta])) \in K'_\infty$.



Let $Z' = K'_\infty \times \prod_{\mathfrak{p}|(\beta)} \overline{\delta_{\mathfrak{f}}(\mathbb{Z}[\beta])}$ be the *stripe space*.

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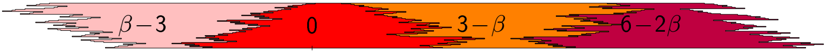

Figure : A patch of $\mathcal{C}_{\mathrm{per}} = \{\delta'(x) + \mathcal{R}(0) : x \in \mathbb{Z}(\beta-3)\}$, $\beta^2 = 2\beta + 2$.

Theorem (M., Steiner 2013)

Let β be a Pisot number. Then the collections $\mathcal{C}_{\mathrm{ext}}$, $\mathcal{C}_{\mathrm{aper}}$, and $\mathcal{C}_{\mathrm{int}}$ are multiple tilings of $\mathbb{K}_\beta$, $\mathbb{K}'_\beta$, and $\mathbb{K}'_\infty$, respectively, and they all have the same covering degree. The following statements are equivalent:

1. All collections $\mathcal{C}_{\mathrm{ext}}$, $\mathcal{C}_{\mathrm{aper}}$, and $\mathcal{C}_{\mathrm{int}}$ are tilings.
2. One of the collections $\mathcal{C}_{\mathrm{ext}}$, $\mathcal{C}_{\mathrm{aper}}$, and $\mathcal{C}_{\mathrm{int}}$ is a tiling.
3. One of the collections $\mathcal{C}_{\mathrm{ext}}$, $\mathcal{C}_{\mathrm{aper}}$, and $\mathcal{C}_{\mathrm{int}}$ has an exclusive point.
4. Property (W) holds.
5. The spectral radius of the boundary graph is less than β.

If (QM) holds, then the following statement is also equivalent to the ones above:

6. $\mathcal{C}_{\mathrm{per}}$ is a tiling of Z'.

Thanks for the attention!