Rauzy fractals and tilings

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Substitution dynamical systems

A *substitution* is an endomorphism of the free monoid $\mathcal{A}^*$, where $\mathcal{A}$ is a finite alphabet. **Tribonacci substitution**: $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$, example of *irreducible unit Pisot* substitution

$$\beta \quad \text{root of} \quad \det(xI - M_\sigma) = x^3 - x^2 - x - 1$$

We study the symbolic dynamical system (X_σ, S) generated by a Pisot substitution σ:

$$X_\sigma = \overline{\{S^n \mathrm{u} \mid n \in \mathbb{N}\}}$$

where $\mathrm{u} \in \mathcal{A}^{\mathbb{N}}$ is a fixed point of σ and S is the shift.

(X_σ, S) is minimal, uniquely ergodic and has entropy zero.

Rauzy 1982: For the Tribonacci substitution (X_σ, S) is measurably conjugate to a minimal toral translation $(\mathbb{T}^2, \tau)$.

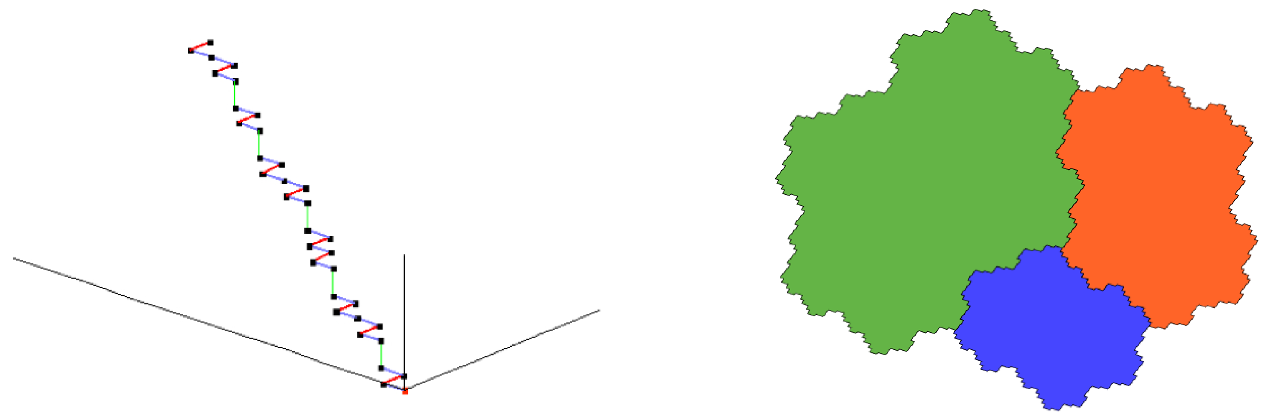
Pisot conjecture

For irred. Pisot substitutions (X_σ, S) has pure discrete spectrum.

Rauzy fractals

Pisot condition: expanding direction $\oplus$ contracting hyperplane.

$$\mathrm{u} = \sigma^\infty(1) = 1213121121312121312112 13 \cdots \in \mathcal{A}^{\mathbb{N}}$$



Rauzy fractal:

$$\mathcal{R}_a = \overline{\{\pi_c \circ P(u_0 \cdots u_{n-1}) \mid n \in \mathbb{N}, u_n = a\}}, \quad \mathcal{R} = \bigcup_{a \in \mathcal{A}} \mathcal{R}_a$$

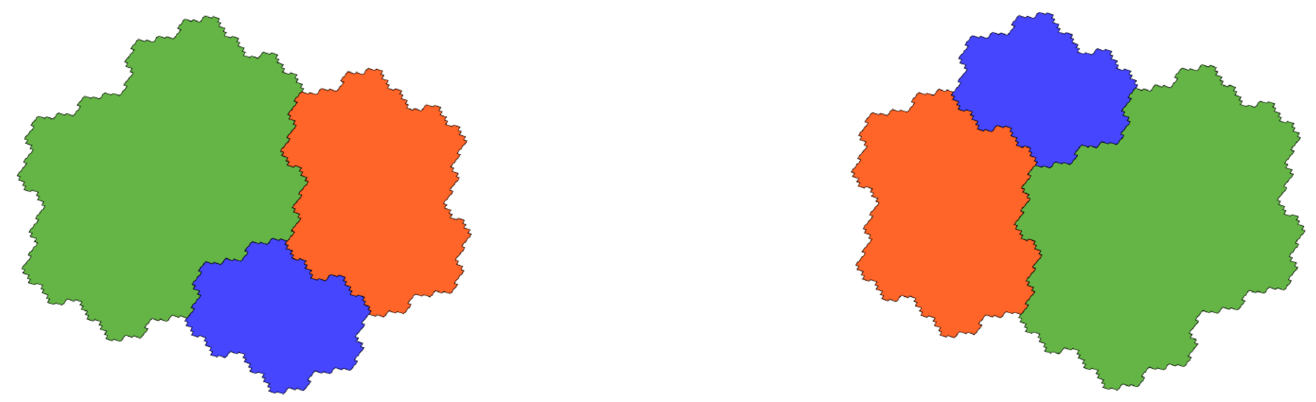
Connection with *Dumont-Thomas* and *beta numeration*. Points of the broken line correspond to β-integers $\sum_{i \geq 0} d_i \beta^i$ (Thurston 1989).

Properties of Rauzy fractals: $\mathcal{R}(x) = \overline{\cup_{k \geq 0} \delta'(\beta^k T_\beta^{-k}(x))}$

- are compact with non-zero Haar measure.
- are the closure of their interior.
- have fractal boundary with zero Haar measure.
- are self-similar (GIFS): $\mathcal{R}(x) = \cup_{y \in T^{-1}(x)} \beta \mathcal{R}(y)$
- $\mathcal{C}_{\text{aper}} = \{\mathcal{R}(x) : x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)\}$ is an aperiodic multiple tiling.

Geometrical approach

How does the action of S on X_σ translate on $\mathcal{R}$?

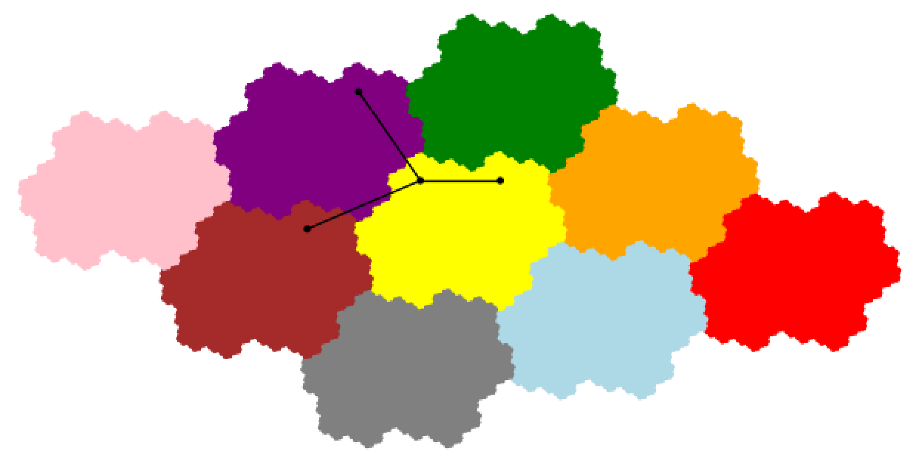


The Strong Coincidence Condition (SCC) holds: the subtiles are disjoint in measure.

Domain exchange transformation:

$$E : \mathcal{R} \to \mathcal{R}, \quad \mathbf{z} \mapsto \mathbf{z} + \pi_c(\mathbf{e}_a) \quad \text{for} \quad \mathbf{z} \in \mathcal{R}_a$$

- The action of E on $\mathcal{R}$ is coded by (X_σ, S) thanks to the partition given by the subtiles.
- Consider the lattice $\Lambda = \sum_{a \in \mathcal{A}} \mathbb{Z}(\pi_c(\mathbf{e}_a) - \pi_c(\mathbf{e}_1))$: all translation directions of E are identified.



- If $\mathcal{C}_{\text{per}} = \mathcal{R} + \Lambda$ is a **tiling** of $\mathbb{C}$, then we have the conjugation with the toral translation $(\mathbb{C}/\Lambda, \tau)$.

This geometrical approach can be extended to any irred. unit Pisot substitution (Arnoux, Ito 2001).

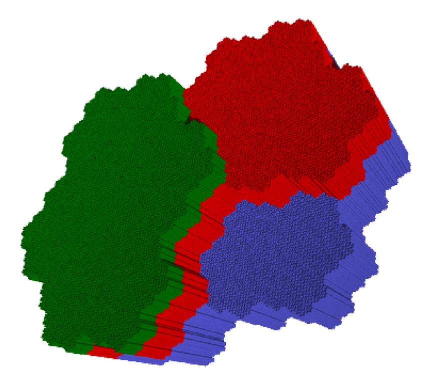
Natural extension

Suspend the Rauzy fractals with intervals (numeration helps):

$$\mathcal{X} = \bigcup_{i=1}^{m-1} [v_i, v_{i+1}) \times (\delta'(v_i) - \mathcal{R}(v_i)),$$

$$\mathcal{T}_\beta : \mathcal{X} \to \mathcal{X}, \quad (x, \mathbf{y}) \mapsto \big(T_\beta(x), \beta \cdot \mathbf{y} - \delta'(\lfloor \beta x \rfloor)\big)$$

- $(\mathcal{X}, \mathcal{B}, \mu, \mathcal{T}_\beta)$ is a natural extension of $([0,1), B, \mu \circ \pi^{-1}, T_\beta)$.
- $\mathbb{K}_\beta = \mathcal{X} + \delta(\mathbb{Z}[\beta^{-1}])$. Tiling $\to$ Markov partition for toral automorphisms.
- $x \in \mathrm{Pur}(\beta)$ if and only if $x \in \mathbb{Q}(\beta)$, $\delta(x) \in \mathcal{X}$.



Beyond irreducibility and unimodularity

- In the reducible case Pisot conjecture is false (Barge, Baker, Kwapisz 2006)! Easy example: Thue-Morse.
- No known example of β-substitution failing the Pisot conjecture.
- Problems with periodic tiling due to $\#\mathcal{A} > \deg(\beta)$. Domain exchange related to the first return to $\mathcal{R}$ under a toral translation.

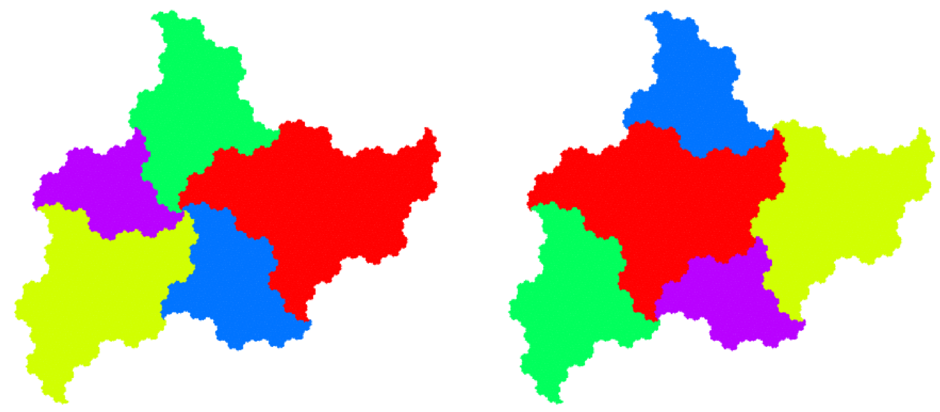


Figure 1: Domain exchange on $\mathcal{R}$ for the minimal Pisot number $\beta^3 = \beta + 1$.

- If β is not a unit we enlarge the representation space so that β becomes a unit therein. Let $K = \mathbb{Q}(\beta)$.

Representation space:

$$\mathbb{K}_\beta := K_\infty \times \prod_{\mathfrak{p} \mid (\beta)} K_{\mathfrak{p}} \subset \mathbb{A}_K$$

where $K_\infty = \mathbb{R}^r \times \mathbb{C}^s$, abs. values given by Galois embeddings, $K_{\mathfrak{p}}$ finite extension of $\mathbb{Q}_p$, for $\mathfrak{p} \mid (p)$, $|\cdot|_{\mathfrak{p}} = \mathfrak{N}(\mathfrak{p})^{-v_{\mathfrak{p}}(\cdot)}$.
$\mathbb{K}_\beta = K_{\mathfrak{p}_1} \times \mathbb{K}'_\beta$ unstable-stable decomposition under mult. by β, δ, δ' diagonal embeddings of K.

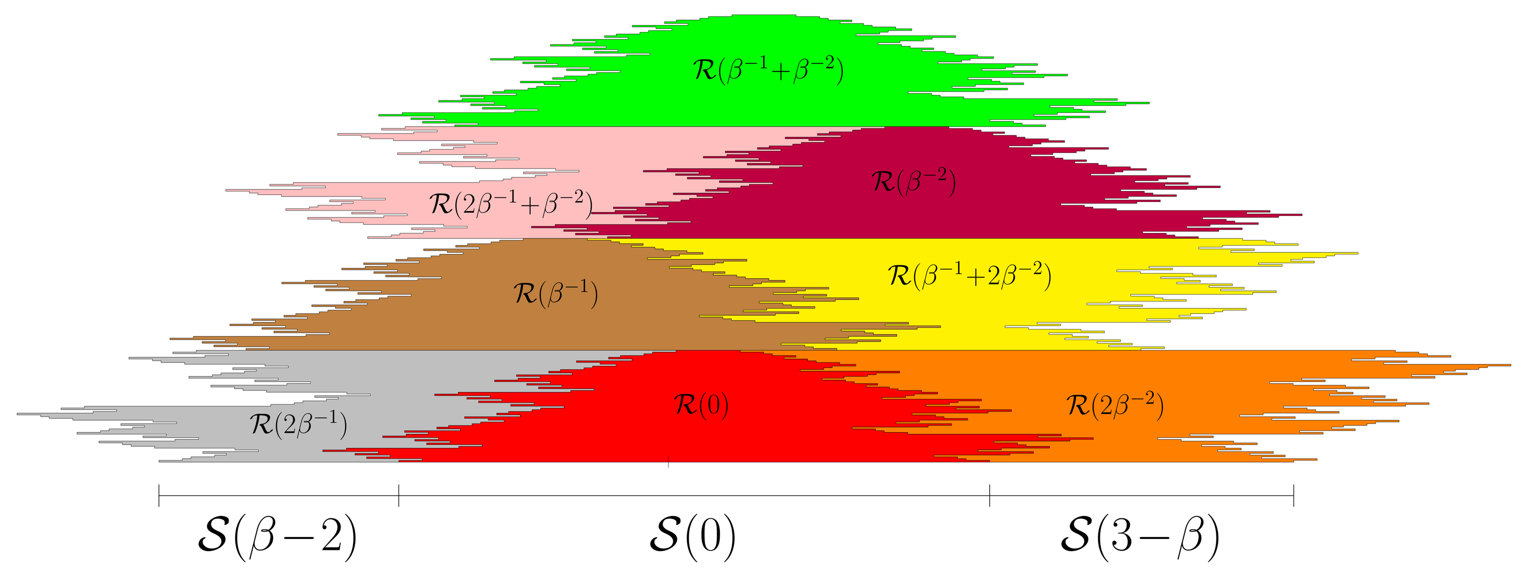


Figure 2: Patch $\beta^{-2}\mathcal{R}(0)$ of $\mathcal{C}_{\text{aper}}$ and integral beta-tiles, $\beta^2 = 2\beta + 2$.

Tilings

Theorem (M., Steiner 2013) In the β-numeration context, β Pisot, $\mathcal{C}_{\text{ext}}$, $\mathcal{C}_{\text{aper}}$, $\mathcal{C}_{\text{int}}$ and $\mathcal{C}_{\text{per}}$ are multiple tilings with same covering degree. They are all tilings provided one of them is a tiling. A spectral condition on a certain boundary graph and a weak finiteness property (W) are necessary and sufficient conditions to get tilings.