Escaping unimodularity for Pisot numeration

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January 14, 2013

- *The geometry of non-unit Pisot substitutions*, with J. Thuswaldner.
- Work in progress, with W. Steiner.

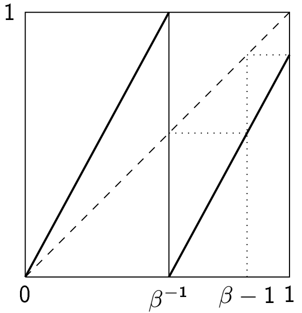

Figure : T_β for $\beta^3 = \beta^2 + \beta + 1$.

Let $\beta > 1$ be a real number. Define

$$\begin{aligned} T_\beta : [0,1) &\to [0,1) \\ x &\mapsto \beta x - \lfloor \beta x \rfloor \end{aligned}$$

Every real $x \in [0,1]$ has a β-*expansion*:

$$(x)_\beta = .d_1 d_2 d_3 \cdots$$

with $d_i \in \mathcal{A} = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$.

$([0,1), T_\beta)$ is conjugate to a *subshift*, the admissibility depending on $(1)_\beta$.

For Pisot β, the subshift is either sofic or of finite type.

Framework: β Pisot number.

Consider the number field $K = \mathbb{Q}(\beta)$ and the finite set of places $S = S_\infty \cup \{\mathfrak{p} : \mathfrak{p} \mid (\beta)\}$. The *representation space* is

$$K_S := K_\infty \times \prod_{\mathfrak{p}|(\beta)} K_\mathfrak{p} = \prod_{\mathfrak{p}\in S} K_\mathfrak{p}$$

where

- $K_\infty = K \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$.
- $K_\mathfrak{p}$ finite extension of $\mathbb{Q}_p$, for $\mathfrak{p} \mid (p)$.

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Cut out the first (expanding) place: $K_{S\setminus\{\mathfrak{p}_1\}}$. Here $\times\beta$ is a contraction!

Embed K into K_S, $K_{S\setminus\{\mathfrak{p}_1\}}$ diagonally by δ, δ'.

The *x-tiles*

For $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$,

$$\mathcal{T}(x) = \overline{\bigcup_{k\geq 0} \delta'(\beta^k T_\beta^{-k}(x))} \in K_{S\setminus\{\mathfrak{p}_1\}}$$

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Cut and project scheme:

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_{\mathfrak{p}_1}} & K_S & \xrightarrow{\pi} & K_{S\setminus\{\mathfrak{p}_1\}} \\ \cup & & \cup & & \cup \\ \mathbb{Z}[\beta^{-1}] & \overset{1-1}{\longleftrightarrow} & \delta(\mathbb{Z}[\beta^{-1}]) & \overset{1-1}{\longleftrightarrow} & \delta'(\mathbb{Z}[\beta^{-1}]) \end{array}$$

- $\delta(\mathbb{Z}[\beta^{-1}])$ is a lattice in K_S.
- $\delta'(\mathbb{Z}[\beta^{-1}] \cap [0,1))$ is a Delone set in $K_{S\setminus\{\mathfrak{p}_1\}}$.

Rauzy fractals: $\mathcal{T}(x)$

- are compact with non-zero Haar measure.
- are the closure of their interior.
- their fractal boundary has zero Haar measure.
- are self-similar (IFS).
- provide a multiple tiling of $K_{S\setminus\{\mathfrak{p}_1\}}$.
- under some conditions provide a tiling.

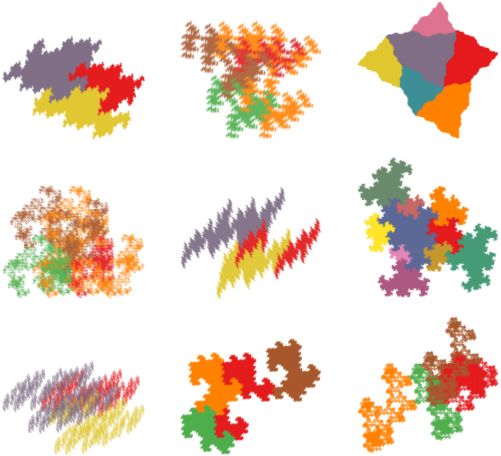

Figure : Euclidean Rauzy fractals by T. Jolivet

Let $\beta^2 = 2\beta + 2$. Representation space: $\mathbb{R} \times K_{(\beta)} \cong \mathbb{R} \times \mathbb{Q}_2^2$.

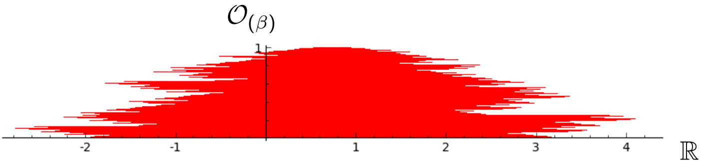

Figure : $\mathcal{T}(0)$ for $\beta^2 = 2\beta + 2$.

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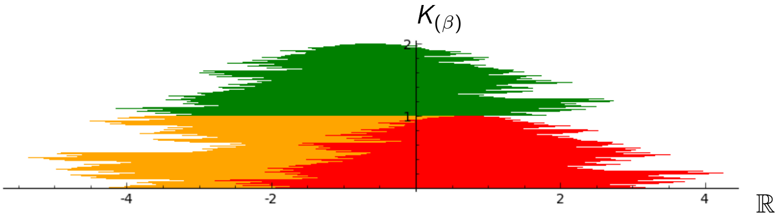

Figure : $\beta^{-1}\mathcal{T}(0)$ for $\beta^2 = 2\beta + 2$.

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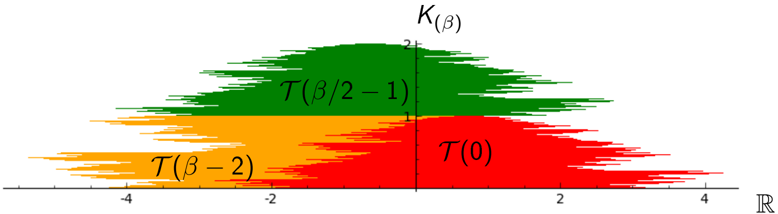

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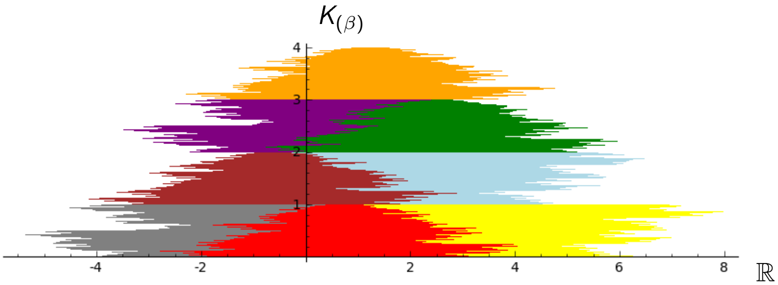

Figure : $\beta^{-2}\mathcal{T}(0)$ for $\beta^2 = 2\beta + 2$.

Let $V = \{v_1, \ldots, v_m\}$ with the $v_i \in \{T_\beta^k(1) : k \geq 0\} \cup \{0\}$ ordered increasingly. Define

$$\mathcal{X} = \bigcup_{i=1}^{m-1} [v_i, v_{i+1}) \times (\delta'(v_i) - \mathcal{T}(v_i)),$$

$$\widetilde{T}_\beta : \mathcal{X} \to \mathcal{X}, \quad (x, \mathbf{y}) \mapsto \big(T_\beta(x), \beta \cdot \mathbf{y} - \delta'(\lfloor \beta x \rfloor)\big)$$

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Theorem

$(\mathcal{X}, \widetilde{\mathscr{B}}, \mu, \widetilde{T}_\beta)$ is a natural extension of $([0,1), \mathscr{B}, \mu \circ \pi^{-1}, T_\beta)$.

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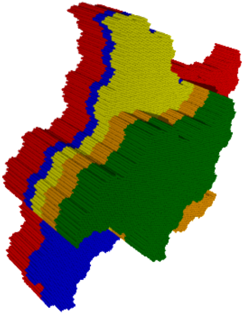
Purely periodic expansions ([Ito, Rao '06], [Berthé, Siegel '07])

$x \in [0,1)$ has a purely periodic β-expansion iff $\delta(x) \in \mathcal{X}$.

Smallest Pisot number natural extension in $\mathbb{R} \times \mathbb{C}$:



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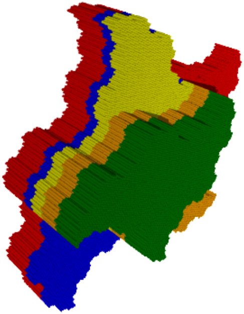

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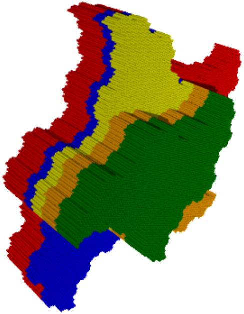

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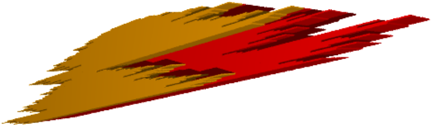

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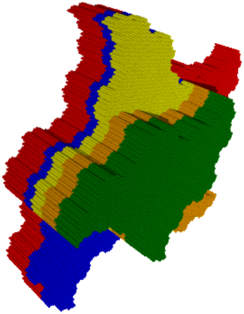

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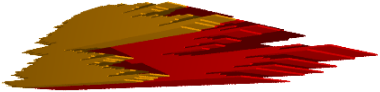

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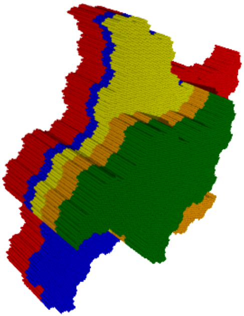

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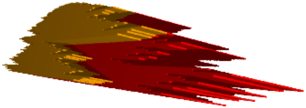

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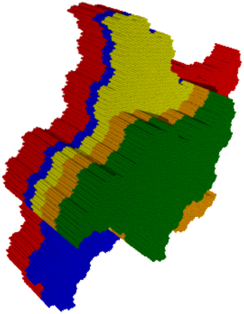

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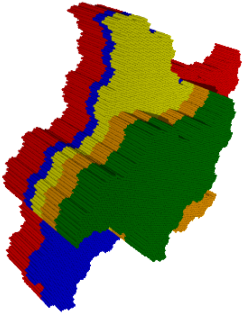

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What we want is that these natural extensions are conjugate to toral/solenoidal automorphisms!

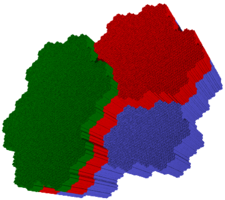
Equivalent conditions for tiling

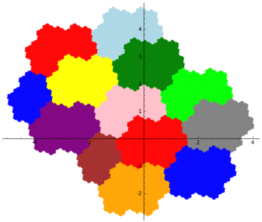
Inspired by [Ito, Rao 2006]:

Theorem

The following are equivalent:

- $\mathrm{cl}(\mathcal{X}) + \delta(\mathbb{Z}[\beta^{-1}])$ forms a tiling of K_S.
- $\{\mathcal{T}(x) : x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)\}$ forms an aperiodic tiling of $K_{S\setminus\{\mathfrak{p}_1\}}$.





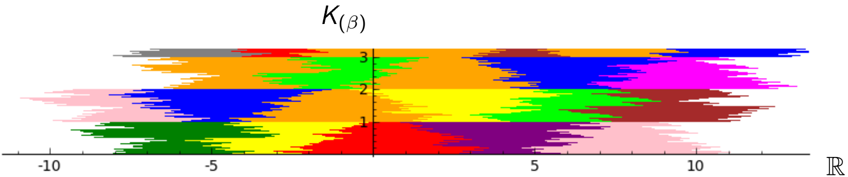
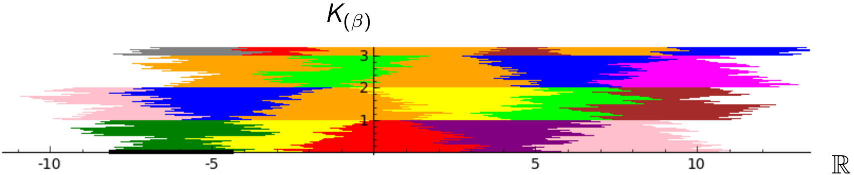


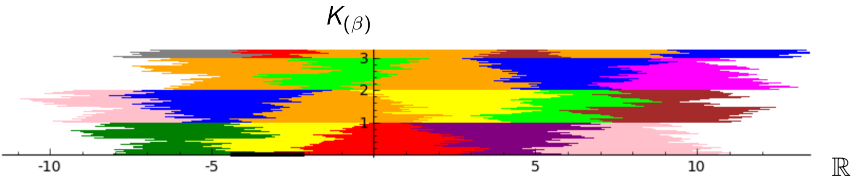
Figure : (Multiple) Tiling of $\mathbb{R} \times K_{(\beta)}$ induced by $\beta^2 = 2\beta + 2$.



Integral β-tiles

For $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$,

$$\mathcal{S}(x) = \{(z_{\mathfrak{p}})_{\mathfrak{p} \in S \setminus \{\mathfrak{p}_1\}} \in \mathcal{T}(x) : z_{\mathfrak{p}} = 0 \ \text{ for each } \ \mathfrak{p} \mid (\beta)\}$$



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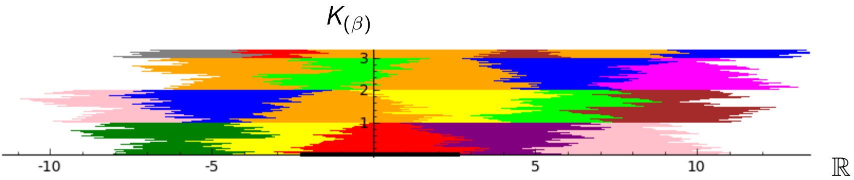

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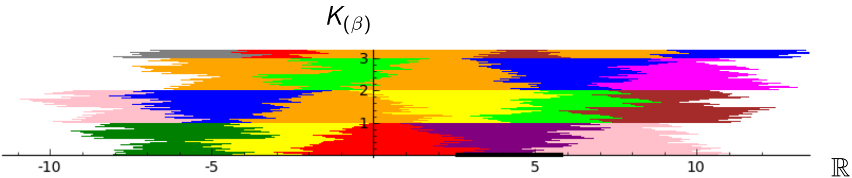

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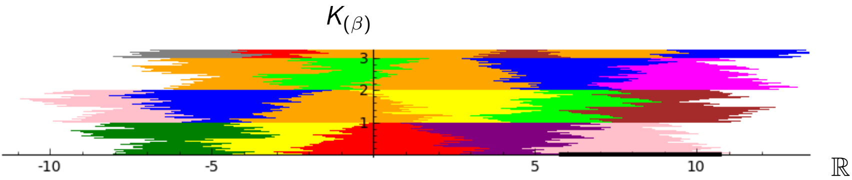

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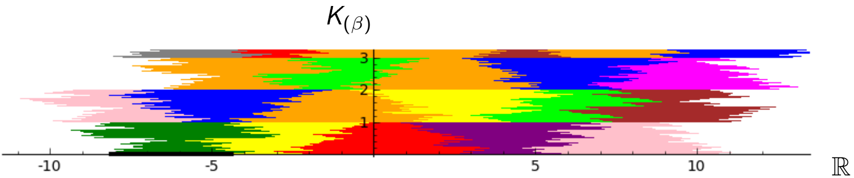

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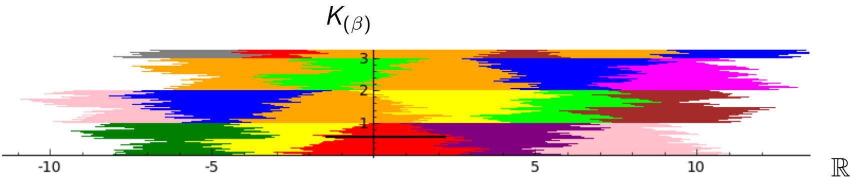

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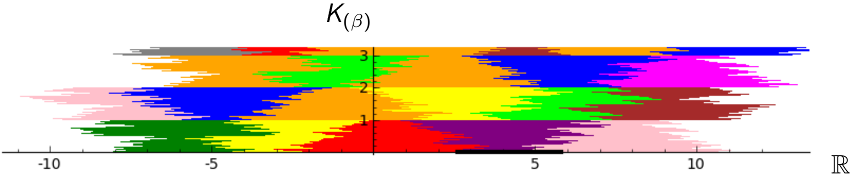

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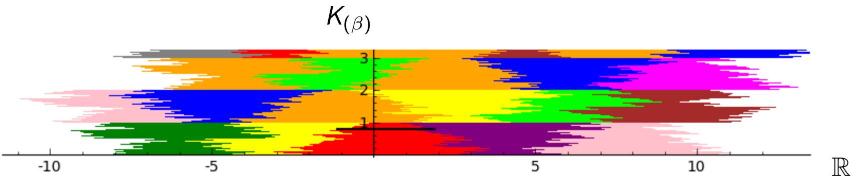

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5. $\mathcal{S}(x)$ are SRS tiles ([Berthé, Siegel, Steiner et al., 2011]).
6. $\{\mathcal{S}(x) : x \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)\}$ forms a weak m-tiling of K'_∞.

Theorem [M., Steiner 201?]

The following are equivalent:

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Back to equivalent conditions for tiling

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$\rightarrow$ Integral β-tiles provide an easy proof that for **quadratic Pisot numbers** the statements above hold!

Property (W):

$$\forall\, y \in P,\ \exists\, z \in \mathbb{Z}[\beta^{-1}] \cap [0,\varepsilon),\ k \geq 0 :\ T_\beta^k(y+z) = T_\beta^k(z) = 0$$

For β Pisot number, $\gamma(\beta)$ is the supremum of the $r \in [0,1]$ such that all $\frac{p}{q} \in \mathbb{Q} \cap (0, r]$ with $\gcd(q, N(\beta)) = 1$ have a purely periodic β-expansion.

- $\gamma(\beta)$ is deeply related with the *boundary graph* ([Akiyama, Barat, Berthé, Siegel 2008]).
- It is possible to describe this graph with integral β-tiles.

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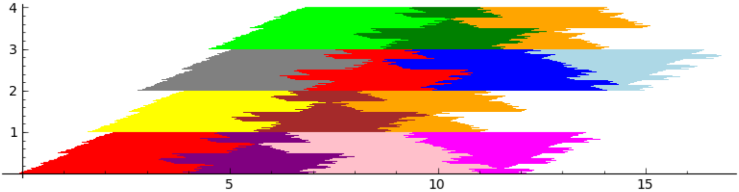
Case: $\beta^2 = a\beta + b$, $a \geq b \geq 1$.
Additional condition: $\gcd(a,b) = 1 \;\Rightarrow\; \mathbb{Q}$ dense in $\prod_{\mathfrak{p}|(\beta)} K_{\mathfrak{p}}$.

Then

$$\gamma(\beta) = \max\left(0, \frac{1 - b - (a - b + 1)\bar{\beta}}{1 - \bar{\beta}^2}\right)$$



Thanks for the attention!