GRAZ UNIVERSITY OF TECHNOLOGY
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# 7. exercise sheet for Mathematics for advanced materials science

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(first name) (last name)

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(student id number)

**7.1.** *(Solving systems of linear equations with a parameter)* (4 credits)

For $x \in \mathbb{R}$, consider the matrix $A_x = \begin{pmatrix} x-1 & -4 \\ -2 & x-3 \end{pmatrix} \in \mathbb{R}^{2\times 2}$.

(a) Find *all* values of $x$ such that the system of linear equations given by $A_x \vec{v} \overset{!}{=} \binom{0}{0}$ admits a solution $\vec{v} \in \mathbb{R}^2$ different from the zero vector.
(Hint: one can deduce from Cramer's rule that it suffices to consider the $x$ such that $\det A_x = 0$.)

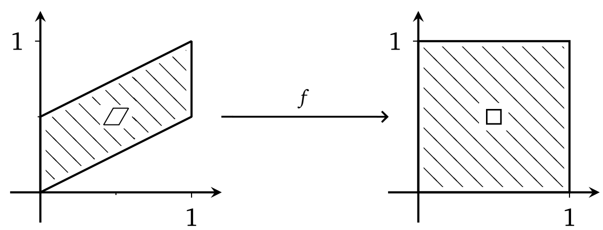
Please submit your solutions digitally at the corresponding TeachCenter course. The deadline is 25.11.2021, 12:15 o'clock. https://tc.tugraz.at/main/course/view.php?id=3543
https://www.math.tugraz.at/~mtechnau/teaching/2021-w-mams.html

(b) For each $x$ determined above, provide a non-zero solution $\vec{v}$ to the above system.

**7.2.** *(Finding certain linear maps)* (4 credits)

Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that the associated linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{v} \mapsto A\vec{v}$, maps the parallelogram ▱ $= \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le 2y - x \le 1\}$ onto the unit square $\square = [0,1] \times [0,1]$, i.e., $f($▱$) = \square$:

$$A = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}.$$

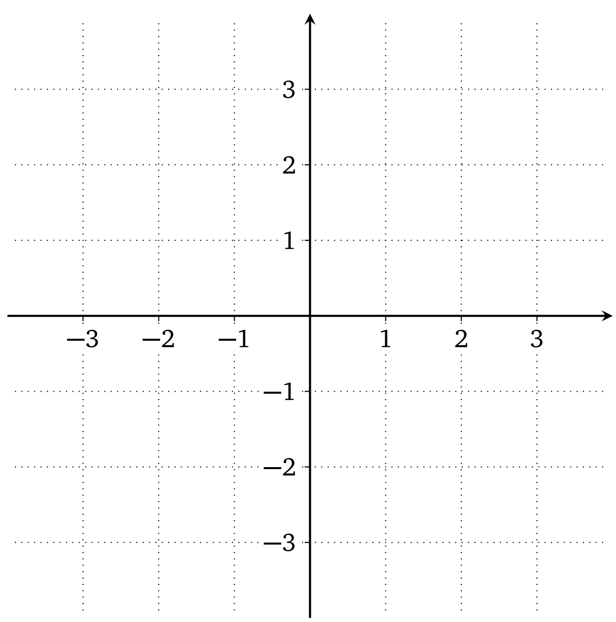



(Hint: it may be easier to find a matrix $B \in \mathbb{R}^{2\times 2}$ such that the associated linear map maps $\square$ onto ▱. One may then take $A = B^{-1}$.)

**7.3.** *(Gram determinants)* (4 credits)

Consider the matrix $A = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \mathbb{R}^{2\times 1}$ and the associated linear map $f : \mathbb{R}^1 \to \mathbb{R}^2$, $v \mapsto Av$.

(a) Sketch the image $\operatorname{im} f = \{f(v) : v \in \mathbb{R}\} \subseteq \mathbb{R}^2$ of $f$ below:



(b) In your above sketch, mark the part of $\operatorname{im} f$ that is $\{f(v) : 0 \le v \le 1\}$ and determine its length.

$$\text{Length of } \{f(v) : 0 \le v \le 1\} = \boxed{\phantom{xxxx}}.$$

(c) Compute $\sqrt{\det(A^{\mathrm{T}}A)} = \boxed{\phantom{xxxx}}$ and $\sqrt{\det(AA^{\mathrm{T}})} = \boxed{\phantom{xxxx}}$.

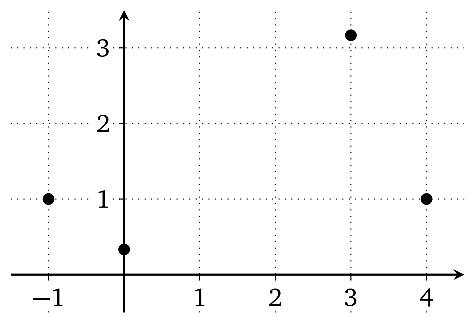
**7.4.** *(Gram determinants)* (4 credits)

Consider the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \in \mathbb{R}^{4\times 2}$ and the vector $\vec{b} = \begin{pmatrix} 1 \\ 1/3 \\ 19/6 \\ 1 \end{pmatrix} \in \mathbb{R}^4$.

(a) Solve the system of linear equations $A^{\mathrm{T}}A\vec{x} \stackrel{!}{=} A^{\mathrm{T}}b$ for $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$.

$$x_1 = \boxed{\phantom{xxxx}}, \quad x_2 = \boxed{\phantom{xxxx}}.$$

(b) With your solution $\vec{x}$ from above, sketch the graph of the affine map $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto x_1 + x_2 t$, below:




(The black points are $(-1, 1)$, $(0, 1/3)$, $(3, 19/6)$ and $(4, 1)$.)

(c) Using the function $f$ from the previous exercise, compute

$$\mathscr{E}_f := (1 - f(-1))^2 + (1/3 - f(0))^2 + (19/6 - f(3))^2 + (1 - f(4))^2. \qquad (\star)$$

$$\mathscr{E}_f = \boxed{\phantom{xxxxx}}.$$

(Hint: the final solution may look slightly ugly, but it is roughly 3.5.)

(d) Pick a vector $(y_1, y_2) \in \mathbb{R}^2$ other than $\vec{x}$ and compute the quantity in $(\star)$ with $f$ replaced by $g : \mathbb{R} \to \mathbb{R}$, $t \mapsto y_1 + y_2 t$. Also sketch the graph of $g$ in the figure in (b).

$$\mathscr{E}_g = \boxed{\phantom{xxxxx}}.$$

(Remark: you may consult § 3.2.6 of the lecture notes for some general context on this exercise.)