Graz University of Technology
Institute of Analysis and Number Theory
Marc Technau

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9. exercise sheet for Mathematics for advanced materials science

(first name) (last name)

(student id number)

9.1. *(Vectors and angles)* (4 credits)

Consider the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(v_1, v_2) \mapsto (-v_2, v_1)$.

(a) Check which of the following statements are true. (None, one or multiple of them may be true.)

- ◯ Geometrically, f describes a rotation by 90° in clockwise direction.
- ◯ Geometrically, f describes a rotation by 90° in anti-clockwise direction.
- ◯ Geometrically, f describes a reflection across the line $\mathbb{R}\binom{0}{1}$.
- ◯ $\operatorname{area} f(\Omega) = \operatorname{area} f(\Omega)$, where Ω is the set $[1,2] \times [0,1]$.
- ◯ $\operatorname{area} f(\Omega) = 2 \operatorname{area} f(\Omega)$, where Ω is the set $[1,8] \times [1,8]$.
- ◯ There is a non-zero vector $\vec{b}$ such that $f(\vec{b}) = \vec{0}$.
- ◯ f has an eigenvector $\vec{b} \in \mathbb{R}^2$.

(b) For vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$, compute

$$\begin{pmatrix} | & | \\ -f(\vec{w}) & f(\vec{v}) \\ | & | \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} | & | \\ \vec{v} & \vec{w} \\ | & | \end{pmatrix} = \begin{pmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{pmatrix}.$$

9.2. *(Area of a triangle)* (4 credits)

Compute the area of the two triangles with the following edges:

(a) $(0,0,0)$, $(1,2,3)$ and $(1,3,3)$ in $\mathbb{R}^3$.

Please submit your solutions digitally at the corresponding TeachCenter course. The deadline is 09.12.2021, 12:15 o'clock. `https://tc.tugraz.at/main/course/view.php?id=3543`
`https://www.math.tugraz.at/~mtechnau/teaching/2021-w-mams.html`

(b) $(0,0,0,0,0,0,0)$, $(1,1,0,2,1,1,1)$ and $(1,3,3,0,1,0,1)$ in $\mathbb{R}^7$.

(Hint: ▲▽. For (b), cross products in $\mathbb{R}^3$ are of no immediate use. However, you should know another approach from earlier lectures.)

Area from (a) = [ ], area from (b) = [ ].

9.3. *(Eigenvalues and eigenvectors, I)* (4 credits)

Consider $(C,n) \in \{(A,2),\ (B,3)\}$, where A and B are the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix},$$

For *both* choices of (C,n) do the following:

(a) determine the characteristic polynomial $\chi_C = \det(X\mathbf{1}_n - C)$ (here "X" should be treated like a variable; think of your favourite number, but do not plug it in),

$\chi_A =$ [ ], $\chi_B =$ [ ],

(b) compute the eigenvalues of C (= the numbers λ that yield zero when substituted for X in the polynomial χ_C) and all associated eigenvectors (= the non-zero solutions $\vec{v} \in \mathbb{R}^n$ of $(\lambda\mathbf{1}_n - C)\vec{v} \overset{!}{=} \vec{0}$),

(c) and discern whether the matrix C is diagonalisable or not (i.e., decide whether you can choose eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ such that the matrix with these eigenvectors as columns has non-zero determinant).

A is diagonalisable: {○ yes, ○ no}, B is diagonalisable: {○ yes, ○ no}.

(Hint: you can find some worked examples in § 3.5 of the lecture notes.)

9.4. *(Eigenvalues and eigenvectors, II)* (4 credits)

Consider the matrix $A \in \mathbb{R}^{2\times 2}$ and the vectors $\vec{b}_1, \dots, \vec{b}_5 \in \mathbb{R}^2$ given below:

$$A = \begin{pmatrix} 11 & -12 \\ 8 & -9 \end{pmatrix}, \quad \vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \vec{b}_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{b}_4 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \vec{b}_5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(a) For each vector $\vec{b}_j$ ($j = 1, \dots, 5$), check whether it is an eigenvector of A and, if it is, determine the corresponding eigenvalue.

j	$\vec{b}_j$ is an eigenvector of A	
1	◯ yes, with associated eigenvalue	☐
	◯ no	
2	◯ yes, with associated eigenvalue	☐
	◯ no	
3	◯ yes, with associated eigenvalue	☐
	◯ no	
4	◯ yes, with associated eigenvalue	☐
	◯ no	
5	◯ yes, with associated eigenvalue	☐
	◯ no	

(b) Let $B_{ij} \in \mathbb{R}^{2\times 2}$ denote the matrix with columns $\vec{b}_i$ and $\vec{b}_j$. Compute the matrix

$$C_{ij} := B_{ij}^{-1} A B_{ij}$$

for all three pairs $(i, j) \in \{(1,3), (1,4), (3,5)\}$.

$$\underbrace{\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}}_{C_{13}}, \quad \underbrace{\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}}_{C_{14}}, \quad \underbrace{\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}}_{C_{35}}.$$