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10. exercise sheet for Mathematics for advanced materials science

10.1. *(Cramer's rule in dimension four)*
Variants of Laplace's expansion hold for arbitrary $n \times n$-matrices. For vectors $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^4$, define their ***cross product*** by

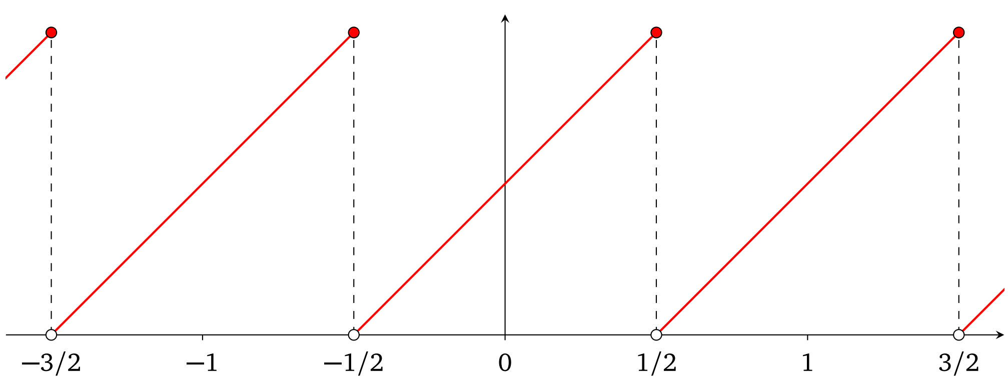
$$\vec{x} \times \vec{y} \times \vec{z} := \det \begin{pmatrix} \vec{e}_1 & | & | & | \\ \vdots & \vec{x} & \vec{y} & \vec{z} \\ \vec{e}_4 & | & | & | \end{pmatrix},$$

where the determinant on the right hand side is supposed to be expanded as in Lemma 3.1 (see also § 3.4 of the lecture notes).

(a) Justify that $\vec{x}, \vec{y}, \vec{z} \perp (\vec{x} \times \vec{y} \times \vec{z})$.

(b) Using the above definition of the cross product, give a 4×4-analogue of Cramer's rule as given in Theorem 3.13. That is, find a formula for the inverse A^{-1} of a matrix $A \in \mathbb{R}^{4\times 4}$ with non-zero determinant, using the columns of A, cross products and dot products.

10.2. *(Fourier series, I)*
Let $g : \mathbb{R} \to \mathbb{R}$ be the 1-periodic function defined by $g(x) = x + 1/2$ for $|x| < 1/2$ and $g(1/2) = 1$. (In particular, $g(-1/2) = g(-1/2 + 1) = g(1/2) = 1$.)



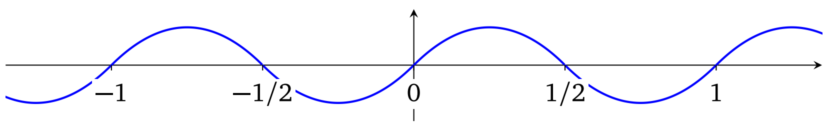
Please submit your solutions digitally at the corresponding TeachCenter course. The deadline is 16.12.2021, 12:15 o'clock. `https://tc.tugraz.at/main/course/view.php?id=3543`
`https://www.math.tugraz.at/~mtechnau/teaching/2021-w-mams.html`

(a) Compute the Fourier coefficients $\hat{g}(k)$ of g for $k \in \mathbb{Z}$. (Hint: the solution can *almost* be found in § 4.3 of the lecture notes. Please consider this exercise as a warm-up for exercise 10.3.)

(b) Determine at which points g is represented by its Fourier series, i.e., for which $x \in \mathbb{R}$ does

$$g(x) = \sum_{k=-\infty}^{\infty} \hat{g}(k) e^{2\pi i k x} \ ?$$

10.3. *(Fourier series, II)*
Let $f : \mathbb{R} \to \mathbb{R}$ be the 1-periodic function defined by $f(x) = (1 - 2|x|)x$ for $|x| \leq 1/2$.



(a) Compute the Fourier coefficients $\hat{f}(k)$ of f for $k \in \mathbb{Z}$. (Hint: this is an exercise in partial integration and requires a bit of tenacity. You may use the following values to verify the validity of your final result: $\hat{f}(1) \approx -0.0645\mathrm{i}$, $\hat{f}(-8) = 0 = \hat{f}(42)$.)

(b) Determine at which points f is represented by its Fourier series, i.e., for which $x \in \mathbb{R}$ does

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x} \ ?$$

(c) Compute $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \pm \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$.
(Hint: use (b) together with a suitably chosen value for x. At the end, you should arrive at a formula for the quantity in question that you can comfortably enter into a calculator. The answer approximately equals 0.9689.)