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11. exercise sheet for Mathematics for advanced materials science

(first name) (last name)

(student id number)

11.1. *(Shifting integral bounds for periodic functions)* (4 credits)
The following text is supposed to provide an answer to a question asked during the last lecture regarding the relation of $\int_0^1$ and $\int_{-1/2}^{1/2}$ when computing Fourier coefficients (cf. exercise 10.2). Fill in the gaps and select the correct statements.

The well-known rule for

○ integration by substitution; ○ partial integration; ○ Fourier analysis

states that for all continuous function $f\colon [a,b] \to \mathbb{R}$ and continuously differentiable $\varphi\colon \ldots$

○ $\varphi\colon [a,b] \to [c,d]$ one has $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_{\varphi(a)}^{\varphi(b)} f(\varphi(y))\varphi'(y)\,\mathrm{d}y$;

○ $\varphi\colon [c,d] \to [a,b]$ one has $\displaystyle\int_{\varphi(c)}^{\varphi(d)} f(x)\,\mathrm{d}x = \int_c^d f(\varphi(y))\varphi'(y)\,\mathrm{d}y$;

○ $\varphi\colon [c,d] \to [a,b]$ one has $\displaystyle\int_a^b f(x)\varphi'(x)\,\mathrm{d}x = \int_{\varphi(c)}^{\varphi(d)} f(\varphi(y))\,\mathrm{d}y$;

○ $\varphi\colon [c,d] \to [a,b]$ one has $\displaystyle\int_{\varphi(c)}^{\varphi(d)} f(\varphi(x))\,\mathrm{d}x = \int_a^b f(y)\varphi'(y)\,\mathrm{d}y$.

Please submit your solutions digitally at the corresponding TeachCenter course. The deadline is 13.01.2022, 12:15 o'clock. https://tc.tugraz.at/main/course/view.php?id=3543
https://www.math.tugraz.at/~mtechnau/teaching/2021-w-mams.html

Upon using this with

$$\varphi(t) = \boxed{}$$

one finds that $\displaystyle\int_{-1/2}^{0} f(x)\,\mathrm{d}x = \int_{1/2}^{1} f(x-1)\,\mathrm{d}x$. Therefore, one has

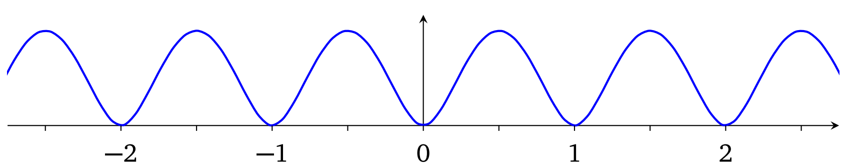
$$\begin{aligned}\int_{-1/2}^{1/2} f(x)\,\mathrm{d}x &= \int_{-1/2}^{0} f(x)\,\mathrm{d}x + \int_{0}^{1/2} f(x)\,\mathrm{d}x = \int_{1/2}^{1} f(x-1)\,\mathrm{d}x + \int_{0}^{1/2} f(x)\,\mathrm{d}x \\ &= \int_{1/2}^{1} f(x)\,\mathrm{d}x + \int_{0}^{1/2} f(x)\,\mathrm{d}x = \int_{0}^{1} f(x)\,\mathrm{d}x\end{aligned}$$

for every continuous function $f : \mathbb{R} \to \mathbb{R}$ that is

◯ 1-periodic; ◯ 2-periodic; ◯ 1/2-periodic; ◯ constant.

(Hint: for the last part, multiple answers may be correct.)

11.2. *(Fourier series, III)* (4 credits)
Let $f : \mathbb{R} \to \mathbb{R}$ be the 1-periodic function defined by $f(x) = x^4 - 2x^3 + x^2$ for $0 \le x < 1$:



(a) Compute the Fourier coefficients $\hat{f}(k)$ of f for $k \in \mathbb{Z}$.
(Hint: depending on how you go about doing this, this requires partial integration four times. You may check your final result using $\hat{f}(0) \approx 1$, $\hat{f}(1) \approx -0.015399$, $\hat{f}(1) \approx -0.015399$, $\hat{f}(-2) \approx -0.00096$.)

$$\hat{f}(0) = \boxed{} \quad \text{and} \quad \hat{f}(k) = \boxed{} \quad (\text{for } k \neq 0).$$

(b) Find complex numbers a_k and b_k such that

$$f(x) = \hat{f}(0) + \sum_{k=1}^{\infty} (a_k \cos(2\pi k x) + b_k \sin(2\pi k x))$$

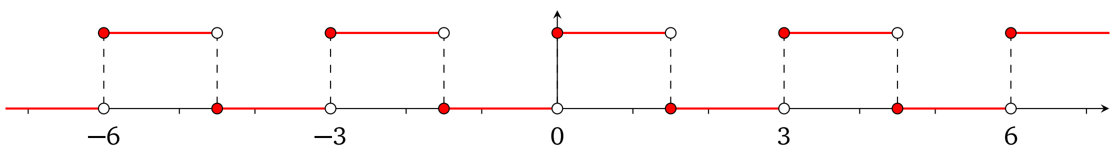
holds for all $x \in \mathbb{R}$.
(Hint: exercise 3.2. Moreover, you can easily test your solution by replacing ∞ in

the sum by 3 [the series in question converges rather quickly], plotting the resulting sum on $[0,1]$ and comparing with a plot of f. They should look almost identical.)

$$a_k = \boxed{} \quad \text{and} \quad b_k = \boxed{}.$$

11.3. *(3-periodic functions)* (4 credits)

Let $g : \mathbb{R} \to \mathbb{R}$ be the 3-periodic function defined by $g(x) = 1$ for $0 \le x < 3/2$ and $g(x) = 0$ for $3/2 \le x < 3$:



(a) What are the values of r for which one might be interested in computing $\hat{g}(r)$? (Hint: "$r \in \mathbb{Z}$" is a *wrong* answer. You should consult Example 4.9 from the lecture notes.)

$$\boxed{}$$

(b) Compute the Fourier coefficients $\hat{g}(r)$ of g for r as in (a).
(Hint: because g is 3-periodic, but not 1-periodic, the Fourier coefficients are *not* given by $\int_0^1 g(x)e^{-2\pi i k x}\,\mathrm{d}x$ which, incidentally, would be zero for all $k \neq 0$. Once you are done, you may compare your answer with the Fourier coefficients $\hat{\chi}(k)$ from Example 4.7.)

$$\hat{g}(0) = \boxed{} \quad \text{and} \quad \hat{g}(r) = \boxed{} \quad (\text{for } r \neq 0).$$

11.4. *(Fourier series in two dimensions)* (4 credits)

Suppose that you are given a (suitably nice) function $f : \mathbb{R}^2 \to \mathbb{R}$ that is known to be periodic with respect to the vectors $\vec{a}_1 = (2,3)$ and $\vec{a}_2 = (1,2)$. Work as in Example 4.10 to find the correct way to write f as a Fourier series:

$$f(\vec{x}) = \sum_{\vec{k}}^{\lightning} \left(\frac{1}{\boxed{?_1}} \int_{\boxed{?_2}} f(\vec{\xi})\, e^{-2\pi i \boxed{?_3}}\, \mathrm{d}^2\vec{\xi} \right) e^{2\pi i \boxed{?_4}}.$$

(Here the sum over $\vec{k}$ ranges over all vectors in $\mathbb{Z}^2$.)

$$\boxed{?_1} = \boxed{}, \quad \boxed{?_2} = \boxed{},$$

$$\boxed{?_3} = \boxed{}, \quad \boxed{?_4} = \boxed{}.$$

(Hint: this turns out to be an exercise in linear algebra rather than Fourier analysis.)