Graz University of Technology
Institute of Analysis and Number Theory
Marc Technau

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12. exercise sheet for Mathematics for advanced materials science

12.1. *(Differentiation)*
Consider $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{x} \mapsto (x_1^2 - x_2, \cos(x_1 x_2))$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$, $\vec{y} \mapsto y_1 + 3y_2^3$. Moreover, let $h\colon \mathbb{R}^2 \to \mathbb{R}$ be given by the composition $h = g \circ f\colon$

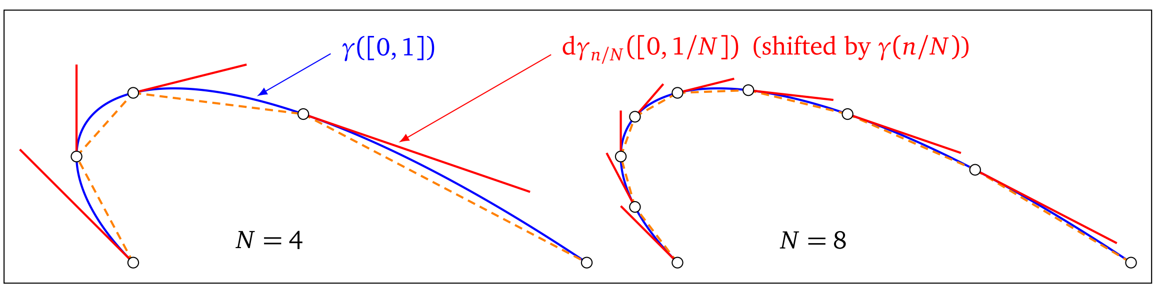
$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "f"'] \arrow[rr, bend right, "h=g\circ f"'] & \mathbb{R}^2 \arrow[r, "g"'] & \mathbb{R} \end{tikzcd}.$$

(a) Compute the matrices $J_f(\vec{x})$, $J_g(\vec{y})$, $J_h(\vec{x})$ and $J_h(f(\vec{x})) \cdot J_f(\vec{x})$.

(b) Argue why f, g and h are differentiable and compute $\mathrm{d}f_{\vec{x}}$, $\mathrm{d}g_{\vec{y}}$ as well as $\mathrm{d}h_{\vec{x}}$.

(c) Compute $\mathrm{d}f_{(1,1)}(4,8)$.

12.2. *(Length of a curve)*
Consider the image $\gamma([0,1]) = \{\gamma(t) : t \in [0,1]\}$ of $[0,1]$ under the function $\gamma\colon \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (2t^2 - t, t - t^3)$. It is a curve in $\mathbb{R}^2$:



(a) Compute $\mathrm{d}\gamma_t\colon \mathbb{R} \to \mathbb{R}^2$.

(b) For $\tau > 0$, compute the length of $\mathrm{d}\gamma_t([0,\tau])$. (Hint: exercise 7.3.)

(c) Compute $\sum_{n=0}^{N-1} \|\gamma((n+1)/N) - \gamma(n/N)\|$ and $\sum_{n=0}^{N-1} \mathrm{length}(\mathrm{d}\gamma_{n/N}([0,1/N]))$ for $N = 4$.

12.3. *(Heat equation)*
Imagine some thin, heated wire spanned between two points which are kept at equal temperature. We model the wire by the interval $[0,1]$ and let $u(t,x)$ denote the temperature of the wire at the point $x \in [0,1]$ and time $t \geq 0$. Abstracting away all units and

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`https://www.math.tugraz.at/~mtechnau/teaching/2021-w-mams.html`

constants of proportionality, the evolution of the resulting function $u\colon \mathbb{R}_{\geq 0} \times [0,1] \to \mathbb{R}$ in this model problem can be seen to be governed by the ***heat equation***

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x)$$

at all points $(t,x) \in \mathbb{R}_{>0} \times (0,1)$.

(a) For $k \in \mathbb{N}$, verify that $x \mapsto \sin(\pi k x)$ is an eigenfunction of the operator mapping infinitely often differentiable functions $\mathbb{R} \to \mathbb{R}$ to their second derivative.

(b) (***Particular solutions***:) Verify that, for every $k \in \mathbb{N}$, the function

$$b_k\colon \mathbb{R}^2 \to \mathbb{R}, \quad (t,x) \mapsto e^{-\pi^2 k^2 t} \sin(\pi k x),$$

satisfies the heat equation, as well as the "boundary condition" $b_k(t,0) = 0 = b_k(t,1)$ for all t and $b_k(0,x) = \sin(\pi k x)$ for all x.

(c) (***Superposition principle***:) Verify that any linear combination $\lambda f + \mu g$ (with numbers λ and μ) of any two functions f, g satisfying the heat equation again satisfies the heat equation.

(d) Let $f : [0,1] \to \mathbb{R}$ be a continuous, piecewise continuously differentiable function with $f(0) = 0 = f(1)$. Show that f can be written as

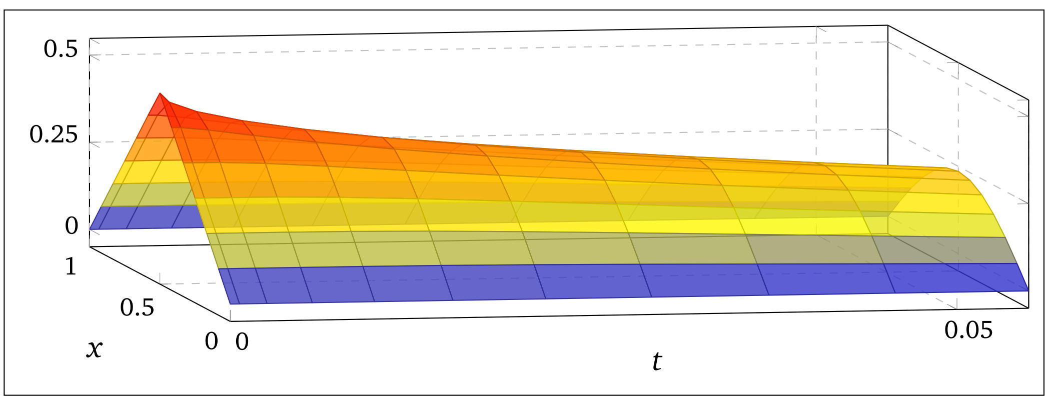
$$f(x) = \sum_{k=1}^{\infty} \tilde{f}(k) \sin(\pi k x) \quad \text{with} \quad \tilde{f}(k) := 2 \int_0^1 f(x) \sin(\pi k x)\, \mathrm{d}x.$$

(e) (***Grand finale***:) Use your insights from all of the above exercises to find an infinite series representing a (the) continuous function $u\colon \mathbb{R}_{\geq 0} \times [0,1] \to \mathbb{R}$ that

- solves the heat equation,
- satisfies the *boundary condition* $u(t,0) = 0 = u(t,1)$ for all t, and
- satisfies the *initial condition* (initial temperature distribution)

$$u(0,x) = (\chi * \chi)(x) \quad \text{for } 0 \leq x \leq 1,$$

where $\chi * \chi$ should be taken from Example 4.8 with parameter $c = 1/2$.



(Hints: (a), (b) and (c) are [meant to be] easy exercises in differentiation. For part (d) try to build a 2-periodic function out of f and deduce the desired result from Theorem 4.1 adapted to 2-periodic functions as in Example 4.9. To find the correct function, recall exercise 3.2. Part (e) may require some partial integration to compute the relevant integral from (d).)