GRAZ UNIVERSITY OF TECHNOLOGY
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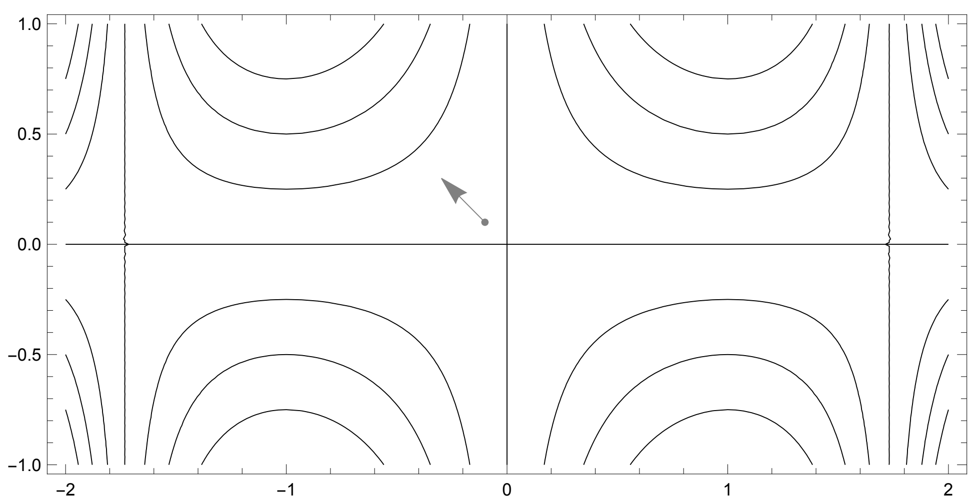
13. exercise sheet for Mathematics for advanced materials science

(first name) (last name)

(student id number)

13.1. *(Gradient fields)* (4 credits)

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^3 y - 3xy$. The figure below shows a ***contour plot*** of f, that is, imagine the graph of f as a mountainous area above the plane $\mathbb{R}^2$ and the drawn lines are the paths on which a hiker would not change his or her altitude.



(a) Compute the gradient $\operatorname{grad} f(x, y) = \begin{pmatrix} \square \\ \square \end{pmatrix}$.

Please submit your solutions digitally at the corresponding TeachCenter course. The deadline is 27.01.2022, 12:15 o'clock. https://tc.tugraz.at/main/course/view.php?id=3543
https://www.math.tugraz.at/~mtechnau/teaching/2021-w-mams.html

(b) For *at least 5 points* $(x,y)\in(-2,2)\times(-1,1)$, compute the gradient $\operatorname{grad} f(x,y)$ and sketch it into in the figure. The base of the arrows you draw should be anchored at (x,y) and not $(0,0)$, however. The grey arrow shows what is meant for the choice $(x,y)=(-1/10,1/10)$. You are not required (but encouraged!) to draw the *length* of the vectors accurately, but the general *direction* of the vectors should be correct.

13.2. *(Polar coordinates, differentiation)* (4 credits)

Consider the function

$$f:\mathbb{R}^2\setminus\{\vec{0}\}\to\mathbb{R},\quad (x,y)\mapsto\frac{2xy}{(x^2+y^2)^2},$$

as well as the well-known polar coordinate map $\vec{P}\colon\mathbb{R}^2\to\mathbb{R}^2$, $(r,\varphi)\mapsto(r\cos\varphi, r\sin\varphi)$. Let $\vec{v}=(1/\sqrt{2},1/\sqrt{2})$. Compute the following quantities.

(a) $\partial_1 f(x,y)=$ ☐ and $\partial_2 f(x,y)=$ ☐.

(b) $\dfrac{\partial f}{\partial\vec{v}}(x,y)=$ ☐.

(c) $(f\circ\vec{P})(r,\varphi)=$ ☐.

(d) $\dfrac{\partial f}{\partial r}(r,\varphi)=$ ☐ and $\dfrac{\partial f}{\partial\varphi}(r,\varphi)=$ ☐.

(Hint: this notation means $\partial_1(f\circ\vec{P})$ and $\partial_2(f\circ\vec{P})$.)

13.3. *(Rotation of vector fields)* (4 credits)

Consider the vector fields $\vec{F}\colon\mathbb{R}^3\to\mathbb{R}^3$, $\vec{x}\mapsto(2x_1,-1,0)$, and $\vec{G}\colon\mathbb{R}^3\to\mathbb{R}^3$, $\vec{x}\mapsto(x_1-x_2, x_2^2x_3, x_3)$.

(a) Compute $\operatorname{rot}\vec{F}=\begin{pmatrix}\square\\\square\\\square\end{pmatrix}$ and $\operatorname{rot}\vec{G}=\begin{pmatrix}\square\\\square\\\square\end{pmatrix}$.

(b) Does there exist a function $f:\mathbb{R}^3\to\mathbb{R}$ such that $\vec{F}=\operatorname{grad} f$? If not, give a quick justification; otherwise exhibit such an f.

(c) Does there exist a function $g\colon \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{G} = \operatorname{grad} g$? If not, give a quick justification; otherwise exhibit such an f.

13.4. *(Heat equation, II)* (4 credits)

Provide a function $u\colon [0,1] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ with the three following properties:

- u solves the heat equation from exercise 13.4 on $(0,1) \times \mathbb{R}_{>0}$.
- $u(t,0) = 0 = u(t,1)$ for all $t \geq 0$.
- $u(0,x) = \sin(2\pi x) - 3\sin(3\pi x)$ for all $x \in [0,1]$.

(Hint: a close look at exercise 13.4 should reveal that solving this exercise does not require any hard calculations or fancy techniques.)