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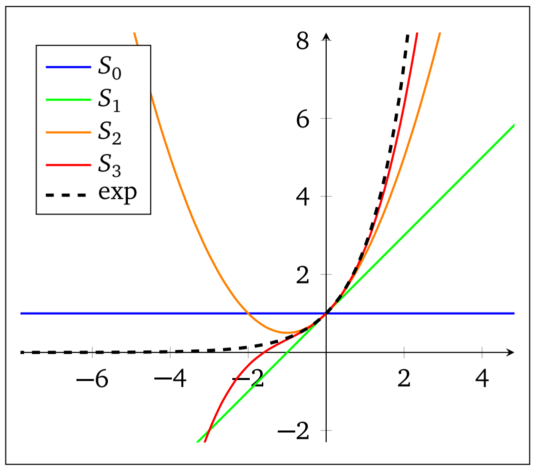
1. exercise sheet for Engineering Mathematics

1.1. *(Partial sums of the exponential function)*
For $n \in \mathbb{N}$ and real x, consider $S_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$ and recall that $\exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$.



(a) Use a calculator to compute $S_n(x)$ for all pairs (n, x) with $n = 0, 1, 2, 3, 4$ and $x = 1, 2$.

(b) Also compute the difference $\exp(x) - S_n(x)$ for the above pairs (n, x)

1.2. *(Geometric series and relatives)*
Suppose that x is a real number such that $|x| < 1$. From the lecture you know that

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}. \tag{†}$$

(a) Differentiate both sides of (†) with respect to x to find a closed-form expression for

$$\sum_{k=0}^{\infty} k x^k. \tag{‡}$$

(You may assume that differentiation commutes with the formation of infinite series, i.e., $\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{\infty} \ldots = \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \ldots$; this is not true in general, but when working with power series it turns out to be fine, provided one stays away from the x for which they diverge.)

(b) Use the above to evaluate (‡) for $x = 1/3$.
(If you get $15/4$ for $x = 3/5$, then your answer to (a) is most likely correct.)

(c) Work as in (a) to find a closed-form expression for

$$\sum_{k=0}^{\infty}(k^2 + k)x^k.$$

1.3. *(Inverse functions)*
Suppose that $f : [0,1] \to [2,3]$ is a bijective function. (This means that for every $y \in [2,3]$ there is one and only one $x \in [0,1]$ such that $f(x) = y$.) Define the inverse function f^{-1} to f to be the function $f^{-1}: [2,3] \to [0,1]$ such that $f^{-1}(f(x)) = x$. The graph G_f of f is defined as the set

$$G_f = \{(x,y) \in [0,1] \times [2,3] : f(x) = y\}$$

and let $G_{f^{-1}}$ be defined similarly. Show that $G_{f^{-1}}$ is obtained from G_f by reflection across the diagonal $\{(x,y) \in \mathbb{R}^2 : x = y\}$.
(Hint: you need to translate what "reflection" means into formulae and ponder a bit what $f^{-1}(f(x)) = x$ actually means.)