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# 9. exercise sheet for Engineering Mathematics

**9.1.** *(Systems of linear differential equations)*
In this exercise, you should apply linear algebra to solve a system of linear differential equations. More precisely, the goal is to find two differentiable functions $x, y : \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \overset{!}{=} \begin{pmatrix} x(t) + 2y(t) \\ 2x(t) + \phantom{2}y(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{†}$$

and

$$x(0) \overset{!}{=} 1, \quad y(0) \overset{!}{=} 3. \tag{‡}$$

(Here a dot above a function means the derivative with respect to $t$, that is, $\dot{x}(t) = x'(t)$.)

(a) Compute the eigenvalues and associated eigenvectors of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

(b) Find an invertible matrix $T \in \mathbb{R}^{2\times 2}$ such that $D = T^{-1}AT$ is a diagonal matrix.

(c) Find differentiable functions $u, v : \mathbb{R} \to \mathbb{R}$ such that, for all $t \in \mathbb{R}$,

$$\begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \end{pmatrix} \overset{!}{=} D \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$$

(Hint: try $t \mapsto \exp(\lambda t)$ for suitable $\lambda$.)

(d) Verify that $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} := T \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ satisfies (†).

(e) Replace your solutions $u$ and $v$ found in (c) with scalar multiples of themselves in such a way that the solution to (†) constructed in (d) also satisfies (‡).

**9.2.** *(Differentiation)*
Consider the two maps $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (xy^2, \exp(x))$, and $g : \mathbb{R}^2 \to \mathbb{R}$, $(v, w) \mapsto v - w$. Compute the following:

(a) $(g \circ f)(x, y)$;

(b) the Jacobian matrices $J_f(x, y)$, $J_g(v, w)$, and $J_{g\circ f}(x, y)$,

(c) the matrix–matrix product $J_g(f(x, y))J_f(x, y)$.

(Hint: examples for computing the Jacobian matrices can be found in § 5.1.3.)