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11. exercise sheet for Engineering Mathematics

11.1. *(Substantial derivative)*
Suppose that $\vec{u}\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ models a time-dependent velocity field. That is, for any point $\vec{x} \in \mathbb{R}^3$ in space and any time $t \in \mathbb{R}$, the vector $\vec{u}(\vec{x}, t)$ is thought of as the velocity of some substance at the given point and time. Imagine an observer floating within the stream given by $\vec{u}$, described by their position function $\vec{X}\colon \mathbb{R} \to \mathbb{R}^3$ ($\vec{X}(t)$ is the position of the observer in space at time t). The position of the observer then satisfies

$$\vec{X}'(t) := J_{\vec{X}}(t) \stackrel{!}{=} \vec{u}(\vec{X}(t), t). \tag{†}$$

Imagine that the observer measures some scalar quantity $c\colon \mathbb{R}^3 \times \mathbb{R}$ (which depends on the position $\vec{x} \in \mathbb{R}^3$ and time t) as they move along their trajectory.

(a) Derive a formula for the rate of change of c *as observed by the observer*.
(Hint: you ought to differentiate $c(\vec{X}(t), t)$ [why?]. Use the chain rule, Theorem 5.6. In fluid dynamics, this is often denoted by $\frac{\mathrm{D}c}{\mathrm{D}t}$ and is called ***substantial derivative*** or ***material derivative***.)

Now put $\vec{u}(\vec{x}, t) = (x_1, 0, 1)$, $\vec{X}(t) = (\exp(t), 5, t)$, and $c(\vec{x}, t) = x_3$.

(b) Verify that $\vec{X}$ and $\vec{u}$ indeed satisfy (†).

(c) Compute $\dfrac{\mathrm{D}c}{\mathrm{D}t}$.

11.2. *(Taylor polynomials)*
For the functions f given below, compute their Taylor polynomials

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

of order $n = 0, 1, 2, 3$ at 0 and use a computer with software of your choice to plot the graphs of these polynomials along with the graph of f on the interval $(-1, 1)$.

(a) $f : (-1, 1) \to \mathbb{R}$, $x \mapsto -\log(1 - x)$;

(b) $f : \mathbb{R} \to (-\pi/2, \pi/2)$, $x \mapsto \arctan(x)$.

(Hint: see § 6.2 in the lecture notes for more on Taylor polynomials. For the task of computing derivatives, you may want to take another look at § 0.2.)