Graz University of Technology
Institute of Analysis and Number Theory
Marc Technau

Winter term 2022
Graz, 20.12.2022

12. exercise sheet for Engineering Mathematics

(first name) (last name)

(student id number)

12.1. *(Taylor polynomials)* (4 credits)

For the functions f given below, compute their Taylor polynomials

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

of order $n = 0, 1, 2, 3$ at 0 ($x_0 = 0$ in the notation from § 6.2) and use a computer with software of your choice to plot the graphs of these polynomials along with the graph of f on the interval $(-1, 1)$.

(a) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^5 + 3x^3 + x - 1$;

$$\sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k =$$

(b) $f : [-1, 1] \to [0, \pi]$, $x \mapsto \arccos(x)$.

$$\sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k =$$

(Hint: see § 6.2 in the lecture notes for more on Taylor polynomials. For the task of computing derivatives, you may want to take another look at § 0.2. It suffices to write down the Taylor polynomials for $n = 3$. For the plots, please use a separate sheet if you cannot superimpose them onto this sheet.)

Please submit your solutions digitally at the corresponding TeachCenter course. The deadline is 10.01.2023, 23:55 o'clock. `https://tc.tugraz.at/main/course/view.php?id=4636`
`https://www.math.tugraz.at/~mtechnau/teaching/2022-w-engimaths.html`

12.2. *(Newton's method)* (4 credits)

In this exercise, we shall use Newton's method to find numerical approximations to the roots of functions. (You can read up on Newton's method in § 6.3 of the lecture notes, but this exercise is self-contained.) Consider the function $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (x\exp(y) - 1, y - x - 1)$.

(a) Compute $J_{\vec{f}}(x,y) = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$.

(b) Determine all $(x,y) \in \mathbb{R}^2$ for which the matrix $J_{\vec{f}}(x,y) \in \mathbb{R}^{2\times 2}$ is invertible and provide a formula for $J_{\vec{f}}(x,y)^{-1}$.

$$J_{\vec{f}}(x,y)^{-1} = \begin{pmatrix} \quad \end{pmatrix} \text{ for all } (x,y) \in \mathbb{R}^2 \text{ such that...}$$

(c) Use your answer for (b) to find a formula for

$$\mathrm{Iter}(x,y) := (x,y) - J_{\vec{f}}(x,y)^{-1}\vec{f}(x,y),$$

assuming that (x,y) are such that $J_{\vec{f}}(x,y)$ is invertible.

$$\mathrm{Iter}(x,y) = \begin{pmatrix} \quad \end{pmatrix} \in \mathbb{R}^2.$$

(d) Set $\vec{x}_0 = (1,1)$, $\vec{x}_1 = \mathrm{Iter}(\vec{x}_0)$, $\vec{x}_2 = \mathrm{Iter}(\vec{x}_1)$, $\vec{x}_3 = \mathrm{Iter}(\vec{x}_2)$. Use your formula from (c) and a calculator (or suitable software) to complete the following table:

i	0	1	2	3
$\vec{x}_i \approx$	$\begin{pmatrix}1\\1\end{pmatrix}$			
$\vec{f}(\vec{x}_i) \approx$				

(Hint: here you are allowed [and encouraged] to use numerical approximations provided by your calculator. Otherwise you get iterated exponentials which quickly become very awkward. For $\vec{f}(\vec{x}_3)$ you should get a vector with quite small entries.)

12.3. *(Polar coordinates, differentiation)* (4 credits)
Consider the function

$$f : \mathbb{R}^2 \setminus \{\vec{0}\} \to \mathbb{R}, \quad (x, y) \mapsto \frac{2xy}{(x^2 + y^2)^2},$$

as well as the well-known polar coordinate map $\vec{P} : \mathbb{R}^2 \to \mathbb{R}^2$, $(r, \varphi) \mapsto (r \cos\varphi, r \sin\varphi)$. Let $\vec{v} = (1/\sqrt{2}, 1/\sqrt{2})$. Compute the following quantities.

(a) $\partial_1 f(x, y) =$ ______ and $\partial_2 f(x, y) =$ ______ .

(b) $\dfrac{\partial f}{\partial \vec{v}}(x, y) =$ ______ .

(c) $(f \circ \vec{P})(r, \varphi) =$ ______ .

(d) $\dfrac{\partial f}{\partial r}(r, \varphi) =$ ______ and $\dfrac{\partial f}{\partial \varphi}(r, \varphi) =$ ______ .

(Hint: this notation means $\partial_1(f \circ \vec{P})$ and $\partial_2(f \circ \vec{P})$.)

12.4. *(Rotation of vector fields)* (4 credits)
Consider the vector fields $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{x} \mapsto (2x_1, -1, 0)$, and $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{x} \mapsto (x_1 - x_2, x_2^2 x_3, x_3)$.

(a) Compute $\operatorname{rot}\vec{F} = \begin{pmatrix} \boxed{\qquad\qquad} \\ \boxed{\qquad\qquad} \\ \boxed{\qquad\qquad} \end{pmatrix}$ and $\operatorname{rot}\vec{G} = \begin{pmatrix} \boxed{\qquad\qquad} \\ \boxed{\qquad\qquad} \\ \boxed{\qquad\qquad} \end{pmatrix}$.

(b) Does there exist a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = \operatorname{grad} f$? If not, give a quick justification; otherwise exhibit such an f.

(c) Does there exist a function $g : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{G} = \operatorname{grad} g$? If not, give a quick justification; otherwise exhibit such an f.

Note: please observe the delayed deadline for submitting solutions due to the winter break.