GRAZ UNIVERSITY OF TECHNOLOGY
INSTITUTE OF ANALYSIS AND NUMBER THEORY
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14. exercise sheet for Engineering Mathematics

____________________ (first name)

____________________ (last name)

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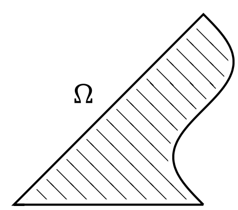
14.1. *(Area computation)* (4 credits)
Compute the area of $\Omega = \{ (x, y) \in \mathbb{R}^3 : 0 \leq x \leq 2\pi - \sin(y),\ 0 \leq y \leq x \}$.

$$\text{area}(\Omega) = \boxed{}.$$



(Hint: see the solution of exercise 13.3.)

14.2. *(Volume of a solid, II)* (4 credits)
Compute the volume of $R = \{ (x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x,\ 0 \leq z \leq xy \}$.

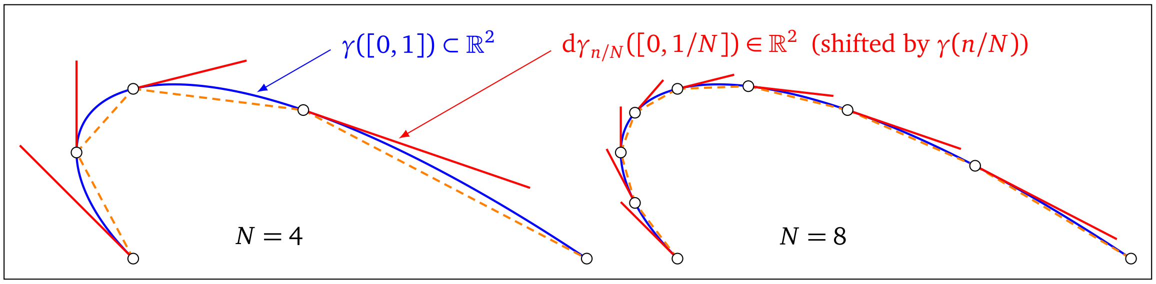
$$\text{vol}(R) = \boxed{}.$$

(Hint: see the solution of exercise 13.3.)

14.3. *(Length of a curve)* (4 credits)
Consider the image $\gamma([0,1]) = \{ \gamma(t) : t \in [0,1] \}$ of $[0,1]$ under the function $\gamma : \mathbb{R} \to \mathbb{R}^2$,

Please submit your solutions digitally at the corresponding TeachCenter course. The deadline is 24.01.2023, 23:55 o'clock. `https://tc.tugraz.at/main/course/view.php?id=4636`
`https://www.math.tugraz.at/~mtechnau/teaching/2022-w-engimaths.html`

$t \mapsto (2t^2 - t, t - t^3)$. It is a curve in $\mathbb{R}^2$:



Let $\tau > 0$ and put $N = 4$. Compute the following:

(a) $d\gamma_t : \mathbb{R} \to \mathbb{R}^2$;

$$d\gamma_t(\tau) = \left(\boxed{} , \boxed{} \right) \in \mathbb{R}^2.$$

(b) the length of $d\gamma_t([0,\tau]) = \boxed{}$;

(c) $\sum_{n=0}^{N-1} \|\gamma((n+1)/N) - \gamma(n/N)\| \approx \boxed{}$;

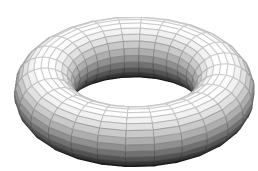
(d) $\sum_{n=0}^{N-1} \text{length}(d\gamma_{n/N}([0,1/N])) \approx \boxed{}$.

(Hints: for (b), see exercise 6.3. For (c) and (d), numerical approximations suffice; you may use a calculator for obtaining these.)

14.4. *(Area of a torus)* (4 credits)

Let $R > r > 0$ and put $U = [0, 2\pi)^2$. The map

$$\vec{\Phi} : U \to \mathbb{R}^3, \quad \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} R\cos(u) + r\cos(u)\cos(v) \\ R\sin(u) + r\sin(u)\cos(v) \\ r\sin(v) \end{pmatrix},$$



parametrises a torus $T = \vec{\Phi}(U)$ (also known as a "doughnut") with outer radius R and inner radius r. Compute the following:

(a) $J_{\vec{\Phi}}(u,v) = \begin{pmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{pmatrix}.$

(b) $\det(J_{\vec{\Phi}}(u,v)^{\mathrm{T}} J_{\vec{\Phi}}(u,v)) = \boxed{}.$

(Hint: the matrix whose determinant is to be computed should turn out to be a diagonal matrix.)

(c) $\operatorname{area}(T) = \iint_T 1 \, \mathrm{d}A = \int_U \sqrt{\det(J_{\vec{\Phi}}(u,v)^{\mathrm{T}} J_{\vec{\Phi}}(u,v))} \, \mathrm{d}^2(u,v) = \boxed{}.$

(Hint: you may verify your result by checking that, for $r = 1$ and $R = 3$, your formula yields ≈ 118.435.)