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5. exercise sheet for Mathematics for Advanced Materials Science

5.1. *(Solving a system of linear equations)*
Consider the following system of linear equations:

$$\begin{pmatrix} 4 & 0 & 2 & 1 \\ 1 & 0 & 2 & 0 \\ 3 & 5 & 0 & 3 \\ 0 & 2 & 5 & 0 \\ 4 & 5 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}.$$

Find the correct value of n such that the above system makes sense (i.e., such that the matrix–vector product on the left hand side can be computed). Subsequently determine all solutions to the above system.
(Hint: if you know Gauß's algorithm, this exercise should be easy for you. If you do not, then you can read up on Gauß's algorithm in § 3.6 of the lecture notes. If you do not want to read this, then you can also simply write out what the above matrix–vector multiplication means and then solve the resulting system of equations using *ad-hoc* considerations.)

5.2. *(Solving a system of linear equations)*
Find *all* solutions $(x_1, x_2, x_3) \in \mathbb{R}^3$ to the following system of linear equations:

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

(The hint from exercise 5.2 also applies to this exercise.)

5.3. *(Finding a matrix representation)*
For each of the following linear maps f_ν, determine the matrix A_ν representing f_ν.

(a) $f_1 \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto -3x$.

(b) $f_2 \colon \mathbb{R}^4 \to \mathbb{R}^2$, $\vec{x} \mapsto (x_2 - x_1, x_3)$.

(c) $f_3 \colon \mathbb{R}^4 \to \mathbb{R}^4$, $\vec{x} \mapsto (x_1 - x_3, x_2, x_1, x_1 + x_3)$.

(d) $f_4 \colon \mathbb{R}^4 \to \mathbb{R}^4$, $\vec{x} \mapsto \vec{y}$, where the vector $\vec{y}$ is determined from $\vec{x}$ such that the following equation is satisfied for all t

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1 + x_2 t + x_3 t^2 + x_4 t^3) = y_1 + y_2 t + y_3 t^2 + y_4 t^3.$$

5.4. *(Composition of maps)*
Consider the linear maps

$$f : \mathbb{R}^3 \to \mathbb{R}^2,\ \vec{v} \mapsto \begin{pmatrix} v_1 + 2v_2 + v_3 \\ 2v_2 + v_3 \end{pmatrix}, \quad \text{and} \quad g : \mathbb{R}^2 \to \mathbb{R}^3,\ \vec{w} \mapsto \begin{pmatrix} w_1 - w_2 \\ w_2/2 \\ 0 \end{pmatrix}.$$

Compute the following:

(a) $(f \circ g)(\vec{w}) := f(g(\vec{w}))$, and $(g \circ f)(\vec{v}) := g(f(\vec{v}))$,

(b) the matrices A, B, C, D representing f, g, $f \circ g$ and $g \circ f$ respectively,

(c) the matrices AB and BA.

(Hint: in (c) you ought to compute a "matrix–matrix product" which you probably already know how to do from some earlier maths course. If you do not, or simply feel a bit rusty with this, please consult the lecture notes; the figure in § 3.1.3 is all you need. The result you get in (c) should look very familar if you have solved (b).)