GRAZ UNIVERSITY OF TECHNOLOGY
INSTITUTE OF ANALYSIS AND NUMBER THEORY
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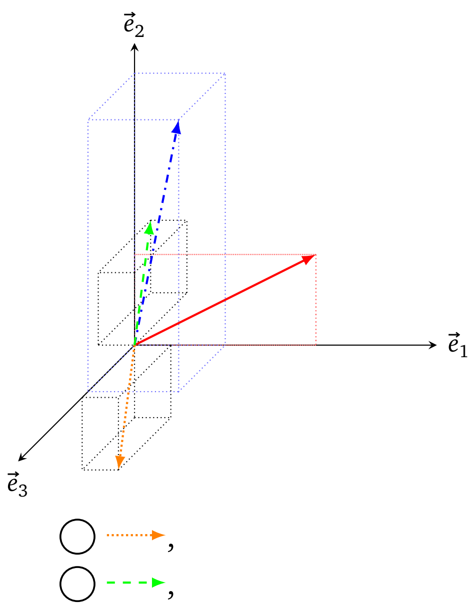
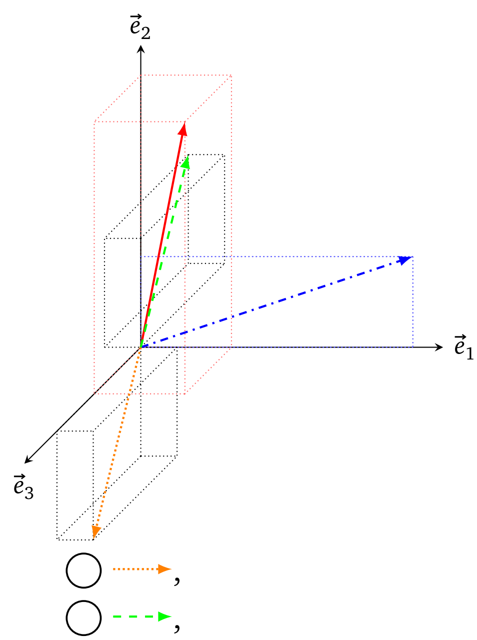
8. exercise sheet for Mathematics for Advanced Materials Science

(first name) (last name)

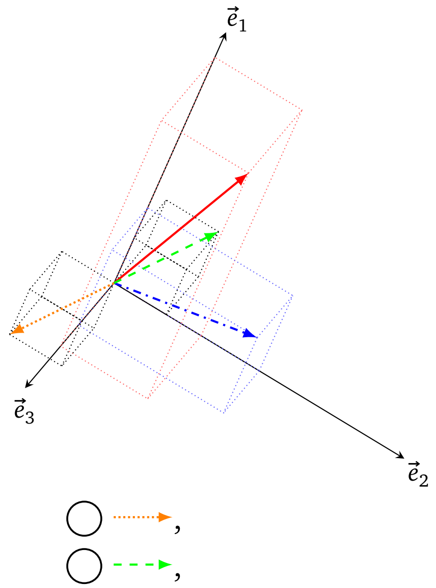
(student id number)

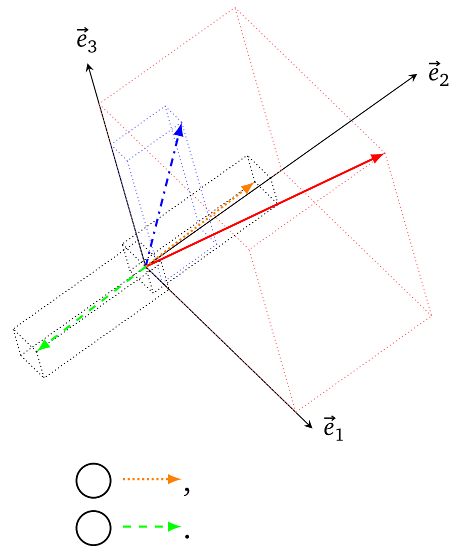
8.1. *(Cross products and orientation)*
In each of the figures below you see a vector $\vec{v}$ drawn as ⟶ and a vector $\vec{w}$ drawn as -·-·→. Discern for each figure whether the vector $\vec{v} \times \vec{w}$ is ⋯→ or - - -→.





Please submit your solutions digitally at the corresponding TeachCenter course. The deadline is 01.12.2022, 23:55 o'clock. `https://tc.tugraz.at/main/course/view.php?id=3543`
`https://www.math.tugraz.at/~mtechnau/teaching/2022-w-mams.html`





(Hint: pay very close attention to the direction of the three standard unit vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ for every figure separately.)

8.2. *(Computing the dot and cross product)* (4 credits)
Consider the three vectors

$$\vec{v}_1 = (0,0,1), \quad \vec{v}_2 = (1,0,2), \quad \vec{v}_3 = (-1,0,2).$$

Compute $\vec{v}_i \bullet \vec{v}_j$ and $\vec{v}_i \times \vec{v}_j$ for all pairs (i,j) of indices with $1 \le i, j \le 3$.
(Hint: *a-priori* there are $2 \cdot 3 \cdot 3 = 18$ things to compute, but by exploiting various symmetries you can reduce your work siginificantly. For instance, $\vec{v}_i \bullet \vec{v}_j = \vec{v}_j \bullet \vec{v}_i$. How do the left and right hand side of this relate when one replaced $\bullet$ by $\times$? Check your answer on $\vec{v}_1 \times \vec{v}_2$ and $\vec{v}_2 \times \vec{v}_1$.)

$$\begin{pmatrix} \vec{v}_1 \bullet \vec{v}_1 & \vec{v}_1 \bullet \vec{v}_2 & \vec{v}_1 \bullet \vec{v}_3 \\ \vec{v}_2 \bullet \vec{v}_1 & \vec{v}_2 \bullet \vec{v}_2 & \vec{v}_2 \bullet \vec{v}_3 \\ \vec{v}_3 \bullet \vec{v}_1 & \vec{v}_3 \bullet \vec{v}_2 & \vec{v}_3 \bullet \vec{v}_3 \end{pmatrix} = \begin{pmatrix} \Box & \Box & \Box \\ \Box & \Box & \Box \\ \Box & \Box & \Box \end{pmatrix},$$

$$\begin{pmatrix} \vec{v}_1 \times \vec{v}_1 & \vec{v}_1 \times \vec{v}_2 & \vec{v}_1 \times \vec{v}_3 \\ \vec{v}_2 \times \vec{v}_1 & \vec{v}_2 \times \vec{v}_2 & \vec{v}_2 \times \vec{v}_3 \\ \vec{v}_3 \times \vec{v}_1 & \vec{v}_3 \times \vec{v}_2 & \vec{v}_3 \times \vec{v}_3 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix}\square\\\square\\\square\end{pmatrix} & \begin{pmatrix}\square\\\square\\\square\end{pmatrix} & \begin{pmatrix}\square\\\square\\\square\end{pmatrix} \\ \begin{pmatrix}\square\\\square\\\square\end{pmatrix} & \begin{pmatrix}\square\\\square\\\square\end{pmatrix} & \begin{pmatrix}\square\\\square\\\square\end{pmatrix} \\ \begin{pmatrix}\square\\\square\\\square\end{pmatrix} & \begin{pmatrix}\square\\\square\\\square\end{pmatrix} & \begin{pmatrix}\square\\\square\\\square\end{pmatrix} \end{pmatrix}.$$

8.3. *(Vectors and angles)* (4 credits)

Consider the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(v_1, v_2) \mapsto (-v_2, v_1)$.

(a) Check which of the following statements are true. (None, one or multiple of them may be true.)

- ◯ Geometrically, f describes a rotation by 90° in clockwise direction.
- ◯ Geometrically, f describes a rotation by 90° in anti-clockwise direction.
- ◯ Geometrically, f describes a reflection across the line $\mathbb{R}\binom{0}{1}$.
- ◯ $\operatorname{area} f(\Omega) = \operatorname{area}\Omega$, where Ω is the set $[1,2] \times [0,1]$.
- ◯ $\operatorname{area} f(\Omega) = 2\operatorname{area}\Omega$, where Ω is the set $[1,8] \times [1,8]$.
- ◯ There is a non-zero vector $\vec{b}$ such that $f(\vec{b}) = \vec{0}$.
- ◯ f has an eigenvector $\vec{b} \in \mathbb{R}^2$.

(b) For vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$, compute

$$\begin{pmatrix} | & | \\ -f(\vec{w}) & f(\vec{v}) \\ | & | \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} | & | \\ \vec{v} & \vec{w} \\ | & | \end{pmatrix} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}.$$

8.4. *(Cramer's rule in four dimensions)*

Variants of Laplace's expansion hold for arbitrary $n\times n$-matrices. For concreteness' sake, we stick to $n = 4$ in this exercise, though. For vectors $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^4$, define their ***cross product*** by

$$\vec{x} \times \vec{y} \times \vec{z} := \det \begin{pmatrix} \vec{e}_1 & | & | & | \\ \vdots & \vec{x} & \vec{y} & \vec{z} \\ \vec{e}_4 & | & | & | \end{pmatrix},$$

where the determinant on the right hand side is supposed to be expanded with respect to the first column as in Lemma 3.1 (see also § 3.4 of the lecture notes); the precise formula cannot be found in the lecture notes, but certainly on the internet. You are welcome to look up the formula if you do not feel comfortable guessing it from your knowledge of the 3×3 case.

(a) Justify that $\vec{x}, \vec{y}, \vec{z} \perp (\vec{x} \times \vec{y} \times \vec{z})$, meaning that $\vec{n} \cdot (\vec{x} \times \vec{y} \times \vec{z}) = 0$ for $\vec{n} \in \{\vec{x}, \vec{y}, \vec{z}\}$.

(b) Using the above definition of the cross product, give a 4×4-analogue of Cramer's rule as given in Theorem 3.13. That is, find a formula for the inverse A^{-1} of a matrix $A \in \mathbb{R}^{4\times 4}$ with non-zero determinant, using the columns of A, cross products and dot products.