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9. exercise sheet for Mathematics for Advanced Materials Science

9.1. *(Eigenvalues and eigenvectors, I)*
Consider $(C,n) \in \{(A,2),(B,3)\}$, where A and B are the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}.$$

For *both* choices of (C,n) do the following:

(a) determine the characteristic polynomial $\chi_C = \det(X\mathbf{1}_n - C)$ (here "X" should be treated like a variable; think of your favourite number, but do not plug it in),

(b) compute the eigenvalues of C (= the numbers λ that yield zero when substituted for X in the polynomial χ_C) and all associated eigenvectors (= the non-zero solutions $\vec{v} \in \mathbb{R}^n$ of $(\lambda\mathbf{1}_n - C)\vec{v} \overset{!}{=} \vec{0}$),

(c) and discern whether the matrix C is diagonalisable or not (i.e., decide whether you can choose eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ such that the matrix with these eigenvectors as columns has non-zero determinant).

(Hint: you can find some worked examples in § 3.5 of the lecture notes.)

9.2. *(Eigenvalues and eigenvectors, II)*
Consider the matrix $A \in \mathbb{R}^{2\times 2}$ and the vectors $\vec{b}_1, \ldots, \vec{b}_5 \in \mathbb{R}^2$ given below:

$$A = \begin{pmatrix} 11 & -12 \\ 8 & -9 \end{pmatrix}, \quad \vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \vec{b}_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{b}_4 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \vec{b}_5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(a) For each vector $\vec{b}_j$ ($j = 1, \ldots, 5$), check whether it is an eigenvector of A and, if it is, determine the corresponding eigenvalue.

(b) Let $B_{ij} \in \mathbb{R}^{2\times 2}$ denote the matrix with columns $\vec{b}_i$ and $\vec{b}_j$. Compute the matrix

$$C_{ij} := B_{ij}^{-1} A B_{ij}$$

for all three pairs $(i,j) \in \{(1,3),(1,4),(3,5)\}$.

9.3. *(Systems of linear differential equations)*
In this exercise, you should apply linear algebra to solve a system of linear differential

equations. More precisely, the goal is to find two differentiable functions $x, y\colon \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \overset{!}{=} \begin{pmatrix} x(t) + 2y(t) \\ 2x(t) + y(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{†}$$

and

$$x(0) \overset{!}{=} 1, \quad y(0) \overset{!}{=} 3. \tag{‡}$$

(Here a dot above a function means the derivative with respect to t, that is, $\dot{x}(t) = x'(t)$.)

(a) Compute the eigenvalues and associated eigenvectors of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

(b) Find an invertible matrix $T \in \mathbb{R}^{2\times 2}$ such that $D = T^{-1}AT$ is a diagonal matrix.

(c) Find differentiable functions $u, v\colon \mathbb{R} \to \mathbb{R}$ such that, for all $t \in \mathbb{R}$,

$$\begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \end{pmatrix} \overset{!}{=} D \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$$

(Hint: try $t \mapsto \exp(\lambda t)$ for suitable λ.)

(d) Verify that $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} := T \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ satisfies (†).

(e) Replace your solutions u and v found in (c) with scalar multiples of themselves in such a way that the solution to (†) constructed in (d) also satisfies (‡).

9.4. *(Evaluation)*
From today until December 15, you have the opportunity to evaluate the present course and provide feedback via TUGRAZonline. Please consider doing so.