GRAZ UNIVERSITY OF TECHNOLOGY
INSTITUTE OF ANALYSIS AND NUMBER THEORY
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Winter term 2022
Graz, 15.12.2022

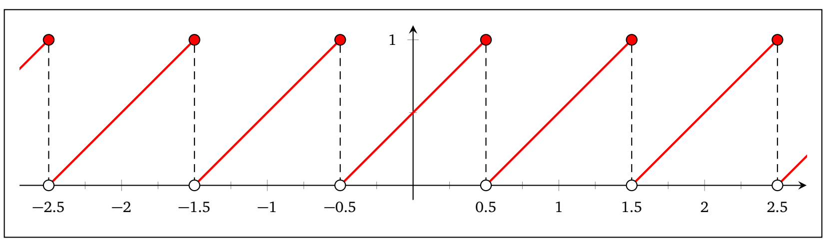
10. exercise sheet for Mathematics for Advanced Materials Science

(first name) (last name)

(student id number)

10.1. *(Fourier series, I)* (4 credits)

Let $g\colon \mathbb{R} \to \mathbb{R}$ be the 1-periodic function defined by $g(x) = x + 1/2$ for $|x| < 1/2$ and $g(1/2) = 1$. (In particular, $g(-1/2) = g(-1/2+1) = g(1/2) = 1$.)



(a) Compute the Fourier coefficients $\hat{g}(k)$ of g for $k \in \mathbb{Z}$. (Hint: $\hat{g}(1) \approx -0.159\mathrm{i}$, $\hat{g}(8) \approx 0.019894\mathrm{i}$. The solution can *almost* be found in § 4.3 of the lecture notes.)

$$\hat{g}(0) = \boxed{} \quad \text{and} \quad \hat{g}(k) = \boxed{} \quad (\text{for } k \neq 0).$$

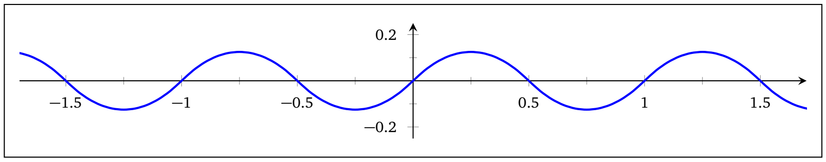
(b) Determine at which points g is represented by its Fourier series, i.e., for which $x \in \mathbb{R}$ does

$$g(x) = \sum_{k=-\infty}^{\infty} \hat{g}(k) e^{2\pi \mathrm{i} k x}\ ? \quad \text{Answer: all } x \in \boxed{}.$$

Please submit your solutions digitally at the corresponding TeachCenter course. The deadline is 12.01.2023, 23:55 o'clock. `https://tc.tugraz.at/main/course/view.php?id=3543`
`https://www.math.tugraz.at/~mtechnau/teaching/2022-w-mams.html`

10.2. *(Fourier series, II)* (4 credits)

Let $f : \mathbb{R} \to \mathbb{R}$ be the 1-periodic function defined by $f(x) = (1 - 2|x|)x$ for $|x| \leq 1/2$.



(a) Compute the Fourier coefficients $\hat{f}(k)$ of f for $k \in \mathbb{Z}$. (Hint: this is an exercise in partial integration and requires a bit of tenacity. You may use the following values to verify the validity of your final result: $\hat{f}(1) \approx -0.0645\mathrm{i}$, $\hat{f}(-8) = 0 = \hat{f}(42)$.)

$$\hat{f}(0) = \boxed{} \quad \text{and} \quad \hat{f}(k) = \boxed{} \quad (\text{for } k \neq 0).$$

(b) Determine at which points f is represented by its Fourier series, i.e., for which $x \in \mathbb{R}$ does

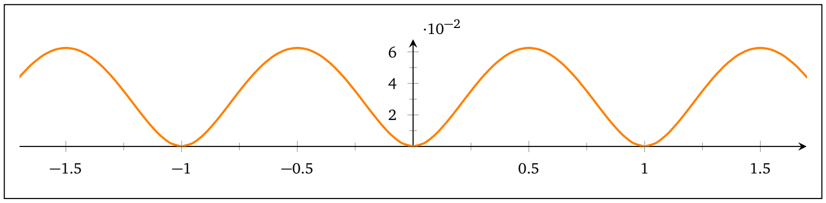
$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi \mathrm{i} k x} \text{ ?} \quad \text{Answer: all } x \in \boxed{}.$$

(c) Compute $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \pm \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \boxed{}$.

(Hint: approximations do not count. Use (b) together with a suitably chosen value for x. At the end, you should arrive at a formula for the quantity in question that you can comfortably enter into a calculator. The answer approximately equals 0.9689.)

10.3. *(Fourier series, III)* (4 credits)

Let $h : \mathbb{R} \to \mathbb{R}$ be the 1-periodic function defined by $h(x) = x^4 - 2x^3 + x^2$ for $0 \leq x < 1$:



(a) Compute the Fourier coefficients $\hat{h}(k)$ of h for $k \in \mathbb{Z}$.
(Hint: depending on how you go about doing this, this requires partial integration four times. You may check your final result using $\hat{h}(0) \approx 0.033$, $\hat{h}(1) \approx -0.015399$, $\hat{h}(1) \approx -0.015399$, $\hat{h}(-2) \approx -0.00096$.)

$$\hat{h}(0) = \boxed{} \quad \text{and} \quad \hat{h}(k) = \boxed{} \quad (\text{for } k \neq 0).$$

(b) Find complex numbers a_k and b_k such that

$$h(x) = \hat{h}(0) + \sum_{k=1}^{\infty}(a_k \cos(2\pi kx) + b_k \sin(2\pi kx))$$

holds for all $x \in \mathbb{R}$.
(Hint: exercise 3.2. Moreover, you can easily test your solution by replacing ∞ in the sum by 3 [the series in question converges rather quickly], plotting the resulting sum on $[0, 1]$ and comparing with a plot of h. They should look almost identical.)

$a_k =$ [ ] and $b_k =$ [ ].

10.4. *(Shifting integral bounds for periodic functions)* (4 credits)

This exercise should show that the integrals $\int_0^1$ and $\int_{-1/2}^{1/2}$ of 1-periodic integrands coincide. Fill in the gaps and select the correct statements.

The well-known rule for integration by ...

◯ substitution; ◯ parts; ◯ Fourier analysis

states that for all continuous function $f : [a, b] \to \mathbb{R}$ and continuously differentiable φ: ...

◯ $\varphi : [a, b] \to [c, d]$ one has $\int_a^b f(x)\,\mathrm{d}x = \int_{\varphi(a)}^{\varphi(b)} f(\varphi(y))\varphi'(y)\,\mathrm{d}y$;

◯ $\varphi : [c, d] \to [a, b]$ one has $\int_{\varphi(c)}^{\varphi(d)} f(x)\,\mathrm{d}x = \int_c^d f(\varphi(y))\varphi'(y)\,\mathrm{d}y$;

◯ $\varphi : [c, d] \to [a, b]$ one has $\int_a^b f(x)\varphi'(x)\,\mathrm{d}x = \int_{\varphi(c)}^{\varphi(d)} f(\varphi(y))\,\mathrm{d}y$;

◯ $\varphi : [c, d] \to [a, b]$ one has $\int_{\varphi(c)}^{\varphi(d)} f(\varphi(x))\,\mathrm{d}x = \int_a^b f(y)\varphi'(y)\,\mathrm{d}y$.

Upon using this with

$\varphi(t) =$ [ ]

one finds that $\int_{-1/2}^{0} f(x)\,\mathrm{d}x = \int_{1/2}^{1} f(x-1)\,\mathrm{d}x$. Therefore, one has

$$\int_{-1/2}^{1/2} f(x)\,\mathrm{d}x = \int_{-1/2}^{0} f(x)\,\mathrm{d}x + \int_{0}^{1/2} f(x)\,\mathrm{d}x = \int_{1/2}^{1} f(x-1)\,\mathrm{d}x + \int_{0}^{1/2} f(x)\,\mathrm{d}x$$
$$= \int_{1/2}^{1} f(x)\,\mathrm{d}x + \int_{0}^{1/2} f(x)\,\mathrm{d}x = \int_{0}^{1} f(x)\,\mathrm{d}x$$

for every continuous function $f : \mathbb{R} \to \mathbb{R}$ that is

◯ 1-periodic; ◯ 2-periodic; ◯ 1/2-periodic; ◯ constant.

(Hint: for the last part, multiple answers may be correct.)

Note: please observe the delayed deadline for submitting solutions due to the winter break.