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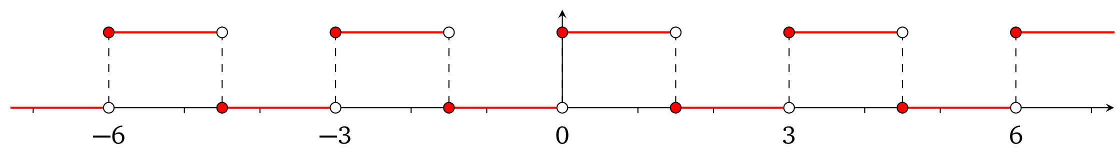
11. exercise sheet for Mathematics for Advanced Materials Science

11.1. *(3-periodic functions)*
Let $g\colon \mathbb{R} \to \mathbb{R}$ be the 3-periodic function defined by $g(x) = 1$ for $0 \leq x < 3/2$ and $g(x) = 0$ for $3/2 \leq x < 3$:



(a) What are the values of r for which one might be interested in computing $\hat{g}(r)$? (Hint: "$r \in \mathbb{Z}$" is a *wrong* answer. You should consult Example 4.9 from the lecture notes.)

(b) Compute the Fourier coefficients $\hat{g}(r)$ of g for r as in (a).
(Hint: because g is 3-periodic, but not 1-periodic, the Fourier coefficients are *not* given by $\int_0^1 g(x)e^{-2\pi i k x}\,\mathrm{d}x$ which, incidentally, would be zero for all $k \neq 0$. Once you are done, you may compare your answer with the Fourier coefficients $\hat{\chi}(k)$ from Example 4.7. Moreover, you may check your result using $\hat{g}(3) \approx -0.1061\mathrm{i}$.)

$\hat{g}(0) =$ ☐ and $\hat{g}(r) =$ ☐ (for $r \neq 0$).

11.2. *(Differentiation)*
Consider the two maps $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (xy^2, \exp(x))$, and $g\colon \mathbb{R}^2 \to \mathbb{R}$, $(v, w) \mapsto v - w$. Compute the following:

(a) $(g \circ f)(x, y)$;

(b) the Jacobian matrices $J_f(x, y)$, $J_g(v, w)$, and $J_{g\circ f}(x, y)$,

(c) the matrix–matrix product $J_g(f(x,y))J_f(x,y)$.

(Hint: examples for computing the Jacobian matrices can be found in § 5.1.3.)

11.3. *(Potentials)*

(a) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $\operatorname{grad} f(x,y) = (2xy-1, x^2)$.

(b) Find a function $g : \mathbb{R}^2 \to \mathbb{R}$ with $\operatorname{grad} g(x,y) = (\sin(x-y)+x\cos(x-y), -x\cos(x-y))$.

(Hint: the gradient of a function is defined in § 5.2 of the lecture notes. To solve this exercise, you just need to know the definition of the gradient, expand it, and see what this tells you about the functions f and g you need to find. Once you have f and g, it is also easy to check that your solution is correct; just compute the gradient.)

11.4. *(Divergence)*

Let $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (F_1(x,y), F_2(x,y))$ be a vector field. Define the ***divergence*** $\operatorname{div} \vec{F}(x,y)$ of $\vec{F}$ at (x,y) to be $\partial_1 F_1(x,y) + \partial_2 F_2(x,y)$ if the appearing partial derivatives exist. Compute $\operatorname{div} \operatorname{grad} f(x,y)$ and $\operatorname{div} \operatorname{grad} g(x,y)$, where f and g are the functions from exercise 11.3.

(Hint: here the main task is to decipher the notation. You can also consult § 6.1 of the lecture notes for a bit more information about the divergence, but you need none of this to solve the present exercise.)