GRAZ UNIVERSITY OF TECHNOLOGY
INSTITUTE OF ANALYSIS AND NUMBER THEORY
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12. exercise sheet for Mathematics for Advanced Materials Science

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(first name) (last name)

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(student id number)

12.1. *(Differentiation)* (4 credits)

Consider the two maps $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (xy, x - y)$, and $g : \mathbb{R}^2 \to \mathbb{R}$, $(v, w) \mapsto v^2 + w^2$. Compute the following:

(a) $(g \circ f)(x, y)$;

(b) the Jacobian matrices $J_f(x, y)$, $J_g(v, w)$, and $J_{g \circ f}(x, y)$,

(c) the matrix–matrix product $J_g(f(x, y))J_f(x, y)$.

Please submit your solutions digitally at the corresponding TeachCenter course. The deadline is 26.01.2023, 23:55 o'clock. `https://tc.tugraz.at/main/course/view.php?id=3543`
`https://www.math.tugraz.at/~mtechnau/teaching/2022-w-mams.html`

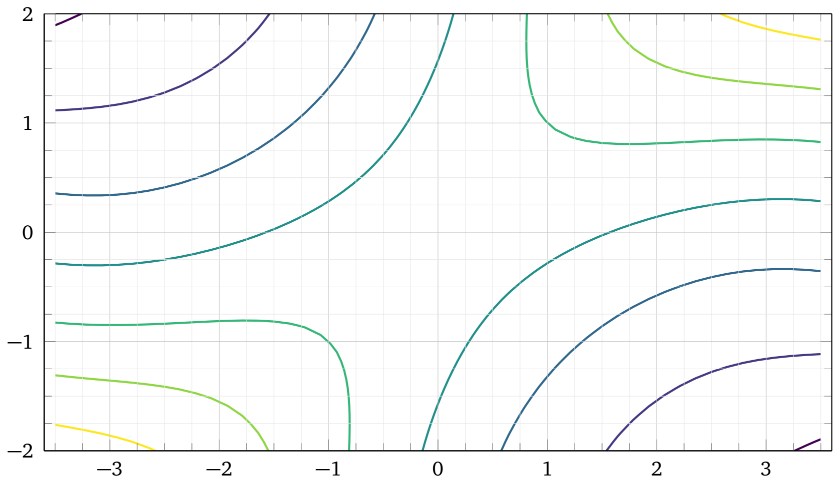
12.2. *(Gradient)* (4 credits)

Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto \cos(x - y) + xy$.

(a) Compute $J_f(x, y)$.

(b) Compute $\operatorname{grad} f(x, y)$.

(c) Pick three distinct points $(x, y) \in [-3, 3] \times [-2, 2]$ for which you compute the gradient $\operatorname{grad} f(x, y)$ numerically and draw it as a vector based at (x, y) in the following picture:



(Hint: the curved lines are curves on which f is constant.)

12.3. *(Polar coordinates, differentiation)* (4 credits)

Consider the function

$$f : \mathbb{R}^2 \setminus \{\vec{0}\} \to \mathbb{R}, \quad (x, y) \mapsto \frac{2xy}{(x^2 + y^2)^2},$$

as well as the well-known polar coordinate map $\vec{P} : \mathbb{R}^2 \to \mathbb{R}^2$, $(r, \varphi) \mapsto (r\cos\varphi, r\sin\varphi)$. Let $\vec{v} = (1/\sqrt{2}, 1/\sqrt{2})$. Compute the following quantities.

(a) $\partial_1 f(x, y) =$ ☐ and $\partial_2 f(x, y) =$ ☐.

(b) $\dfrac{\partial f}{\partial \vec{v}}(x, y) =$ ☐. (Hint: Lemma 5.1)

(c) $(f \circ \vec{P})(r, \varphi) = \boxed{}$.

(d) $\dfrac{\partial f}{\partial r}(r, \varphi) = \boxed{}$ and $\dfrac{\partial f}{\partial \varphi}(r, \varphi) = \boxed{}$.

(Hint: this notation means $\partial_1(f \circ \vec{P})$ and $\partial_2(f \circ \vec{P})$.)

12.4. *(Divergence)* (4 credits)

Let $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (F_1(x, y), F_2(x, y))$ be a vector field. Define the ***divergence*** $\operatorname{div} \vec{F}(x, y)$ of $\vec{F}$ at (x, y) to be $\partial_1 F_1(x, y) + \partial_2 F_2(x, y)$ if the appearing partial derivatives exist. Compute $\operatorname{div} \operatorname{grad} f(x, y)$ and $\operatorname{div} \operatorname{grad} g(x, y)$, where f and g are the functions from exercise 11.3, i.e., $f(x, y) = x^2 y - x + c_f$ and $g(x, y) = x \sin(x - y) + c_g$, for arbitrary constants $c_f, c_g \in \mathbb{R}$.
(Hint: here the main task is to decipher the notation.)