GRAZ UNIVERSITY OF TECHNOLOGY
INSTITUTE OF ANALYSIS AND NUMBER THEORY
Marc Technau

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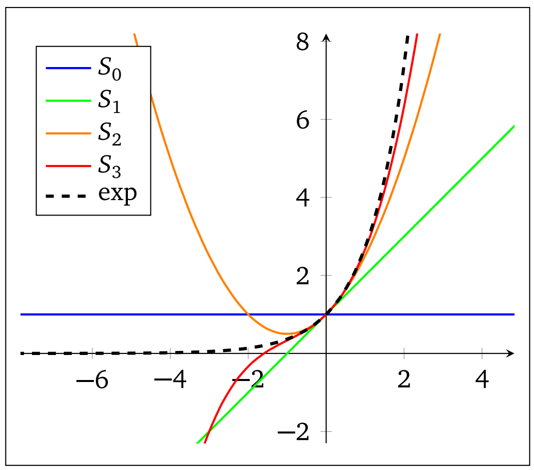
1. exercise sheet for Engineering Mathematics

1.1. *(Partial sums of the exponential function)*

For $n \in \mathbb{N}$ and real x, consider $S_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$ and recall that $\exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$.



(a) Use a calculator to compute $S_n(x)$ for all (n,x) with $n = 0,1,2,3,4$ and $x = -1,0$.

(b) Also compute the difference $\exp(x) - S_n(x)$ for the above pairs (n,x).

1.2. *(Geometric series and relatives)*

Suppose that x is a real number such that $|x| < 1$. From the lecture you know that

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}. \tag{†}$$

(a) Differentiate both sides of (†) with respect to x to find a closed-form expression for

$$\sum_{k=0}^{\infty} k x^k. \tag{‡}$$

(You may assume that differentiation commutes with the formation of infinite series, i.e., $\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{\infty} \ldots = \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \ldots$; this is not true in general, but when working with power series it turns out to be fine, provided one stays away from the x for which they diverge.)

(b) Use the above to evaluate (‡) for $x = 1/3$.
(If you get $15/4$ for $x = 3/5$, then your answer to (a) is most likely correct.)

(c) Work as in (a) to find a closed-form expression for

$$\sum_{k=0}^{\infty}(k^2 + k)x^k.$$

1.3. *(Power series "Ansatz" for differential equations)*
Suppose that $y : \mathbb{R} \to \mathbb{R}$ is a function which satisfies the *differential equation* $y'(x) = y(x)$ for all $x \in \mathbb{R}$ and the "initial condition" $y(0) = 2$. Suppose further that y can be written as a power series:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad (\text{for } x \in \mathbb{R}). \qquad (\star)$$

(a) Find the numeric value of a_0.

(b) As in exercise 1.2 (a), differentiate the power series in $(\star)$ term-wise and use the differential equation for y to find a_1 and a_2. (Hint: as before, just compute with power series as you would with polynomials; we ignore all issues regarding convergence here. Moreover, you may use the fact that a power series equals the zero function precisely if all its coefficients equal zero.)

(c) Continue your work from (b) to find an expression for a_n for all $n \in \mathbb{N}$. Use this to determine y in terms of known functions.