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Winter term 2023
Graz, 18.10.2023

3. exercise sheet for Engineering Mathematics

3.1. *(Integration via substitution)*
Consider the function $f : [-1,1] \to \mathbb{R}$, $x \mapsto \sqrt{1-x^2}$.

(a) What is the distance of the point $(x, f(x))$ to the origin? (Hint: Pythagoras.)

(b) Use your insight from part (a) to sketch the graph of f. What shape does it represent?

(c) The area enclosed in-between the graph of f and the horizontal coordinate axis is given by $I = \int_{-1}^{1} f(x)\,\mathrm{d}x$. Use your insight from part (b) and your knowledge of geometry to find the value of I without much computation.

(d) Now, try to compute I using integration via substitution. (Hint: your geometric insights from (b) should already suggest an appropriate substitution function φ. Subsequently, the trigonometric identity $(\sin t)^2 + (\cos t)^2 = 1$ may be useful.)

3.2. *(Integration, II)*
Compute the following integrals:

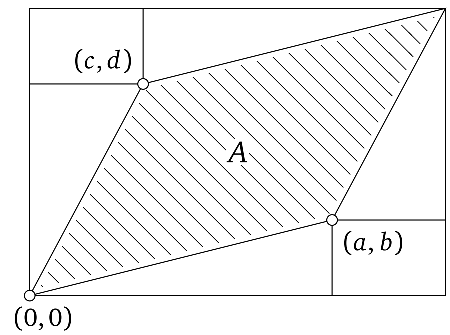
(a) $\displaystyle\int_0^1 2\sin(x)\exp(x)\,\mathrm{d}x,$ (≈ 1.8187 for verification)

(b) $\displaystyle\int_1^2 2\cos(\log(x))\,\mathrm{d}x,$ (≈ 1.8164)

(c) $\displaystyle\int_0^1 \arcsin(x)\,\mathrm{d}x.$ (≈ 0.5708)

(Hints: for (a), use integration by parts, differentiating the relevant trigonometric function and integrating the exponential function each time. You should get an equation where the original integral shows up twice. This can then be solved. For (b), use a substitution to pass to an integral similar to the one treated in part (a). This can then be computed as in (a). For (c), recall that arcsin is the inverse function of sin: $[-\pi/2, \pi/2] \to [-1,1]$. Use a substitution to exploit that $\arcsin(\sin(t)) = t$ for $t \in [-\pi/2, \pi/2]$.)

3.3. *(A problem in analytic geometry)*
Stare at the picture on the right hand corner.



(a) Find a formula for the area $A = A(a,b,c,d)$ of the hatched parallelogram with vertices $(0,0)$, (a,b), $(a+c, b+d)$ and (c,d) in terms of a, b, c and d.

(b) Interchanging the two points (a, b) and (c, d) produces the same parallelogram, thus, $A(a, b, c, d) = A(c, d, a, b)$. Check whether the formula you obtained in part (a) satisfies this equation. You may get three outcomes:

(1) Your formula does satisfy the above. (Bravo!—You were exceptionally careful!)
Try to figure out what a student who gets outcome (2) below may have done differently.

(2) You get $A(a, b, c, d) = -A(c, d, a, b)$ according to your formula.
Modify your formula so that it also gives the correct area in this case.

(3) None of the above: re-do part (a) and then part (b).

(c) Consider the setting of outcome (2) in part (b). (If your outcome was (1), then, having done the corresponding task, you should also know what to do here.) Try to come up with a "rule of thumb" for when your formula for A produces a negative number rather than the expected positive result.

(Hint: the tasks leave a bit of room for interpretation. Do not get frustrated by this, for this exercise will not be graded; there are no credits to be lost here.)