Graz University of Technology
Institute of Analysis and Number Theory
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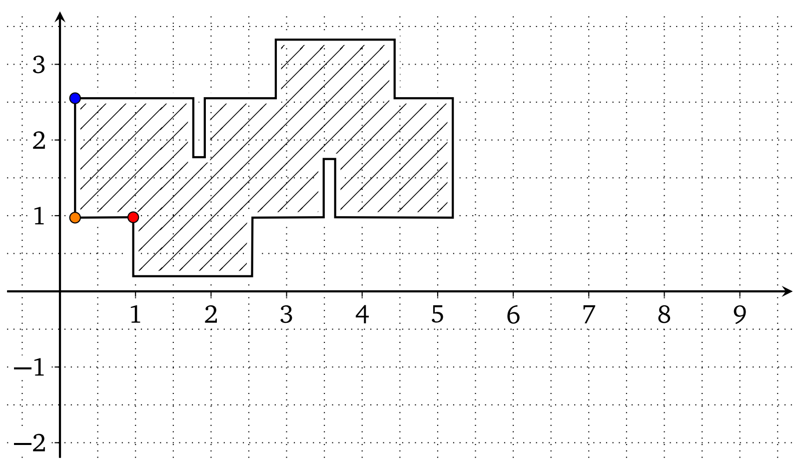
6. exercise sheet for Engineering Mathematics

(first name)	(last name)
(student id number)	

6.1. *(Determinants and areas)* (4 credits)

Consider the shape $\mathscr{T}$ enclosed by connecting the following 20 points $\vec{x}$ in their given order with line segments and connecting back to the starting point:



$(0.970, 0.979)$, $(0.200, 0.974)$,
$(0.200, 2.552)$, $(1.765, 2.552)$,
$(1.765, 1.773)$, $(1.917, 1.773)$,
$(1.917, 2.552)$, $(2.857, 2.552)$,
$(2.857, 3.326)$, $(4.430, 3.326)$,
$(4.430, 2.552)$, $(5.200, 2.552)$,
$(5.200, 0.974)$, $(3.643, 0.979)$,
$(3.643, 1.749)$, $(3.491, 1.749)$,
$(3.491, 0.979)$, $(2.548, 0.974)$,
$(2.543, 0.200)$, $(0.970, 0.200)$.

($\mathscr{T}$ is the hatched region drawn above.) Let A be the matrix $\begin{pmatrix} 1 & 1.5 \\ -0.5 & 1 \end{pmatrix}$. Let $\mathscr{T}'$ be the shape obtained by repeating the above construction of $\mathscr{T}$, but with each of the above points $\vec{x}$ replaced by $A\vec{x}$.

(a) Sketch $\mathscr{T}'$ in the above coordinate system.
(Hint: to get a reasonably accurate figure, you need only compute $A\vec{x}$ for three choices of $\vec{x}$, and certainly not all 20. Do not try to be accurate to every millimetre, but do try to produce an adequate sketch. Please use a ruler, but do not invest more than 10 minutes on the drawing itself.)

Please submit your solutions during the next lecture (22.11.2023).
https://www.math.tugraz.at/~mtechnau/teaching/2023-w-engimaths.html

(b) Suppose that you knew that the area of $\mathscr{T}$ equals 10.0826 (approximately). Find the area of $\mathscr{T}'$.

$$\text{area}(\mathscr{T}') = \boxed{}.$$

(Hint: recall § 3.2.5 from the lecture notes. Again, the computational effort here should be rather low and should certainly not involve computing all of the points $A\vec{x}$.)

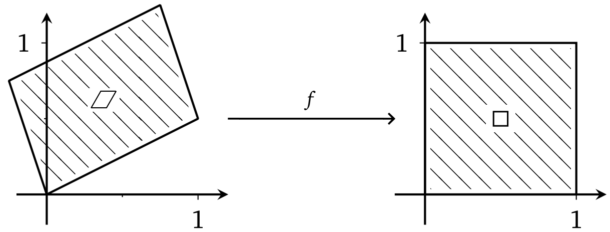
6.2. *(Finding certain linear maps)* (4 credits)

Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that the associated linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{v} \mapsto A\vec{v}$, maps the parallelogram

$$\varparallelogram = \{\,(x, y) \in \mathbb{R}^2 : 0 \le \tfrac{6}{7}x + \tfrac{2}{7}y \le 1,\ 0 \le \tfrac{8}{7}y - \tfrac{4}{7}x \le 1\,\}$$

onto the unit square $\square = [0, 1] \times [0, 1]$, i.e., $f(\varparallelogram) := \{\,f(\vec{v}) : \vec{v} \in \varparallelogram\,\} = \square$:

$$A = \begin{pmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{pmatrix}.$$



(Hint: it may be easier to find a matrix $B \in \mathbb{R}^{2\times 2}$ such that the associated linear map maps $\square$ onto $\varparallelogram$. One may then take $A = B^{-1}$.)

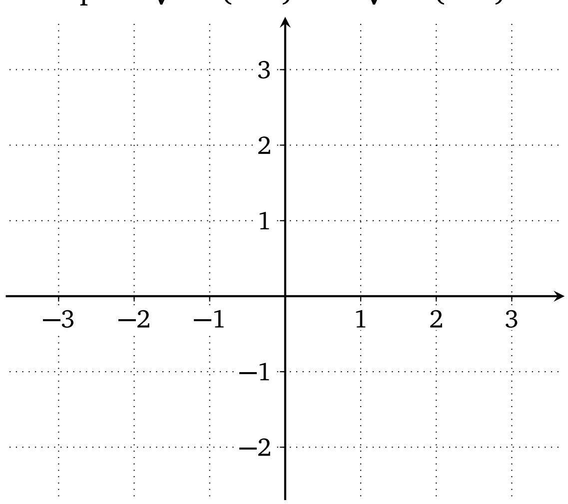
6.3. *(Gram determinants)* (4 credits)

Consider the matrix $A = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \in \mathbb{R}^{2\times 1}$ and the associated linear map $f : \mathbb{R}^1 \to \mathbb{R}^2$, $v \mapsto Av$.

(a) Sketch the image $\operatorname{im} f = \{\,f(v) : v \in \mathbb{R}\,\} \subseteq \mathbb{R}^2$ of f below.

(b) In your above sketch, mark the part of $\operatorname{im} f$ that is $\{\,f(v) : 0 \le v \le 1\,\}$ and determine its length L.

(c) Compute $\sqrt{\det(A^{\mathsf{T}}A)}$ and $\sqrt{\det(AA^{\mathsf{T}})}$.



(b) Length $L = \boxed{}$.

(c) $\sqrt{\det(A^{\mathsf{T}}A)} = \boxed{}$,

$\sqrt{\det(AA^{\mathsf{T}})} = \boxed{}$.

6.4. *(Gram determinants)* (4 credits)

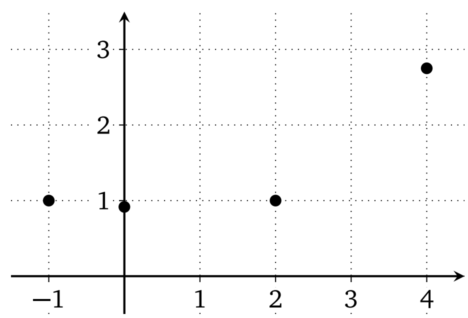
The following exercise illustrates how one may use linear algebra to find curves to data (see § 3.2.6 of the lecture notes for more background information, but reading it is not strictly necessary for solving this exercise). Consider the matrix A and the vector $\vec{b}$ given below:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 4 \end{pmatrix} \in \mathbb{R}^{4\times 2}, \quad \vec{b} = \begin{pmatrix} 1 \\ 11/12 \\ 1 \\ 11/4 \end{pmatrix} \in \mathbb{R}^4.$$

(a) Solve the system of linear equations $A^{\mathrm{T}} A \vec{x} \overset{!}{=} A^{\mathrm{T}} \vec{b}$ for $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$.

$$x_1 = \boxed{}, \quad x_2 = \boxed{}.$$

(b) With your solution $\vec{x}$ from above, sketch the graph of the affine map $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto x_1 + x_2 t$, below:



(The black points are $(-1, 1)$, $(0, 11/12)$, $(2, 1)$ and $(4, 11/4)$.)

(c) Using the function f from the previous exercise, compute

$$\mathcal{E}_f := (1 - f(-1))^2 + (11/12 - f(0))^2 + (1 - f(2))^2 + (11/4 - f(4))^2. \qquad (\star)$$

$$\mathcal{E}_f = \boxed{}.$$

(Hint: the final solution may look slightly ugly, but it is roughly 0.7.)

(d) Pick a vector $(y_1, y_2) \in \mathbb{R}^2$ other than $\vec{x}$ and compute the quantity in $(\star)$ with f replaced by $g : \mathbb{R} \to \mathbb{R}$, $t \mapsto y_1 + y_2 t$. Also sketch the graph of g in the figure in (b).

$$\mathcal{E}_g = \boxed{}.$$