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7. exercise sheet for Engineering Mathematics

7.1. *(Computing the dot and cross product)*
Consider the three vectors

$$\vec{v}_1 = (1,0,1), \quad \vec{v}_2 = (1,1,0), \quad \vec{v}_3 = (-1,0,2).$$

Compute $\vec{v}_i \bullet \vec{v}_j$ and $\vec{v}_i \times \vec{v}_j$ for all pairs (i,j) of indices with $1 \leq i,j \leq 3$.
(Hint: *a-priori* there are $2 \cdot 3 \cdot 3 = 18$ things to compute, but by exploiting various symmetries you can reduce your work siginificantly. For instance, $\vec{v}_i \bullet \vec{v}_j = \vec{v}_j \bullet \vec{v}_i$. How do the left and right hand side of this relate when one replaces $\bullet$ by $\times$? Check your answer on $\vec{v}_1 \times \vec{v}_2$ and $\vec{v}_2 \times \vec{v}_1$.)

7.2. *(Area of a triangle)*
Compute the area of the two triangles with the following vertices:

(a) $(0,0)$, $(1,2)$ and $(1,3)$ in $\mathbb{R}^3$.

(b) $(0,0,0)$, $(1,2,3)$ and $(1,3,3)$ in $\mathbb{R}^3$.

(c) $(0,0,0,0,0,0,0)$, $(1,1,0,2,1,1,1)$ and $(1,2,1,0,1,0,1)$ in $\mathbb{R}^7$.

(Hint: ▲▽. The answers are roughly 0.5, 1.6 and 3.4 respectively.)

7.3. *(Orthogonal matrices)*
A matrix $A \in \mathbb{R}^{n\times n}$ is called ***orthogonal*** if $A^{\mathrm{T}}A = \mathbf{1}_n$. For the sake of concreteness, suppose that $n = 3$ for the remainder of the exercise. Let $\vec{a}_{\bullet 1}, \vec{a}_{\bullet 2}, \vec{a}_{\bullet 3} \in \mathbb{R}^3$ denote the columns of A.

(a) Show that A is orthogonal if and only if

$$\|\vec{a}_{\bullet j}\| = 1 \quad \text{and} \quad \vec{a}_{\bullet j} \bullet \vec{a}_{\bullet k} = 0$$

holds for all $1 \leq j,k \leq 3$, $j \neq k$.
(Hint: write $A = (a_{ij})_{i,j=1}^{3}$ and compute $A^{\mathrm{T}}A$. Now try to see some norms and dot products in there.)

(b) For each of the following three matrices, find values to put into the last column such that the resulting matrix is orthogonal if such values exist. If no such values exist, argue why this is the case.

$$\begin{pmatrix} 1 & 0 & v_1 \\ 0 & -1 & v_2 \\ 0 & 0 & v_3 \end{pmatrix}, \quad \begin{pmatrix} 1/\sqrt{3} & 0 & w_1 \\ -1/\sqrt{3} & 1/\sqrt{2} & w_2 \\ 1/\sqrt{3} & 1/\sqrt{2} & w_3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & z_1 \\ 1 & -1 & z_2 \\ 1 & 1 & z_3 \end{pmatrix}.$$