Graz University of Technology
Institute of Analysis and Number Theory
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Winter term 2023
Graz, 29.11.2023

8. exercise sheet for Engineering Mathematics

(first name) (last name)

(student id number)

8.1. *(Vectors and angles)* (4 credits)

Consider the linear map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(v_1, v_2) \mapsto (v_2, -v_1)$.

(a) Check which of the following statements are true. (None, one or multiple of them may be true. Wrong answers also count negatively, so do not get tempted to check too much.)

- ◯ Geometrically, f describes a rotation by 90° in clockwise direction.
- ◯ Geometrically, f describes a rotation by 90° in anti-clockwise direction.
- ◯ Geometrically, f describes a reflection across the line $\mathbb{R}\binom{0}{1}$.
- ◯ $\operatorname{area} f(\Omega) = \operatorname{area} \Omega$, where Ω is the set $[1,8] \times [1,8]$.
- ◯ $\operatorname{area} f(\Omega) = 2 \operatorname{area} \Omega$, where Ω is the set $[1,2] \times [0,1]$.
- ◯ There is a non-zero vector $\vec{b}$ such that $f(\vec{b}) = \vec{0}$.
- ◯ f has an eigenvector $\vec{b} \in \mathbb{R}^2$.

(b) For vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$, compute

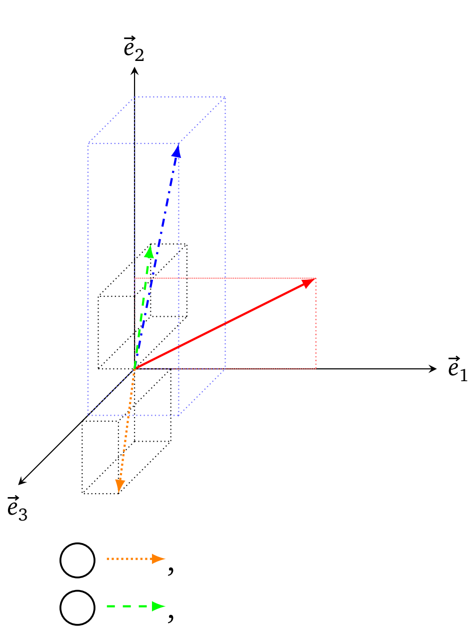
$$\begin{pmatrix} | & | \\ -f(\vec{w}) & f(\vec{v}) \\ | & | \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} | & | \\ \vec{v} & \vec{w} \\ | & | \end{pmatrix} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}.$$

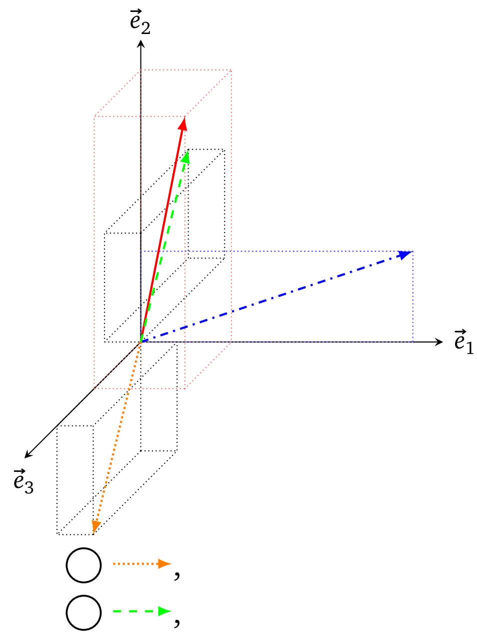
Please submit your solutions during the next lecture (06.12.2023).

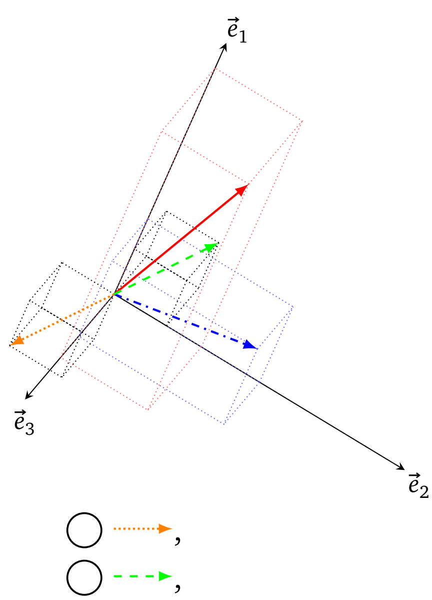
https://www.math.tugraz.at/~mtechnau/teaching/2023-w-engimaths.html

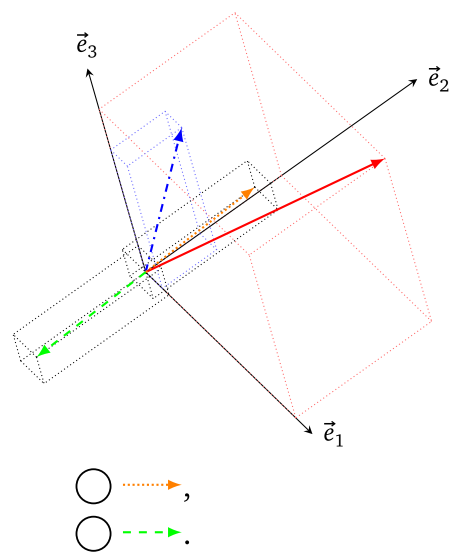
8.2. *(Cross products and orientation)* (4 credits)

In each of the figures below you see a vector $\vec{v}$ drawn as ⟶ and a vector $\vec{w}$ drawn as -·-·→. Discern for each figure whether the vector $\vec{v} \times \vec{w}$ is ⋯⋯→ or - - -→.









(Hint: pay very close attention to the direction of the three standard unit vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ for every figure separately.)

8.3. *(Eigenvalues and eigenvectors, I)* (4 credits)

For this exercise, you should read § 3.5 of the lecture notes if you are not already sufficiently familiar with eigenvalues, eigenvectors and diagonalisability. In particular, you can find some worked examples there. Consider $(C,n) \in \{(A,2), (B,3)\}$, where A and B are the following matrices:

$$A = \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}.$$

For *both* choices of (C,n) do the following:

(a) determine the characteristic polynomial $\chi_C = \det(X\mathbf{1}_n - C)$ (here "X" should be treated like a variable; think of your favourite number, but do not plug it in),

$$\chi_A = \boxed{}, \quad \chi_B = \boxed{},$$

(b) compute the eigenvalues of C (= the numbers λ that yield zero when substituted for X in the polynomial χ_C) and all associated eigenvectors (= the non-zero solutions $\vec{v} \in \mathbb{R}^n$ of $(\lambda\mathbf{1}_n - C)\vec{v} \overset{!}{=} \vec{0}$),

(c) and discern whether the matrix C is diagonalisable or not (i.e., decide whether you can choose eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ such that the matrix with these eigenvectors as columns has non-zero determinant).

A is diagonalisable: $\left\{\begin{matrix} \bigcirc & \text{yes} \\ \bigcirc & \text{no} \end{matrix}\right\}$, B is diagonalisable: $\left\{\begin{matrix} \bigcirc & \text{yes} \\ \bigcirc & \text{no} \end{matrix}\right\}$.

8.4. *(Eigenvalues and eigenvectors, II)* (4 credits)

Consider the matrix $A \in \mathbb{R}^{2\times 2}$ and the vectors $\vec{b}_1, \dots, \vec{b}_5 \in \mathbb{R}^2$ given below:

$$A = \begin{pmatrix} 7 & -6 \\ 4 & -3 \end{pmatrix}, \quad \vec{b}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \vec{b}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{b}_4 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \vec{b}_5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(a) For each vector $\vec{b}_j$ ($j = 1, \dots, 5$), check whether it is an eigenvector of A and, if it is, determine the corresponding eigenvalue.

j	$\vec{b}_j$ is an eigenvector of A	
1	◯ yes, with associated eigenvalue	☐
	◯ no	
2	◯ yes, with associated eigenvalue	☐
	◯ no	
3	◯ yes, with associated eigenvalue	☐
	◯ no	
4	◯ yes, with associated eigenvalue	☐
	◯ no	
5	◯ yes, with associated eigenvalue	☐
	◯ no	

(b) Let $B_{ij} \in \mathbb{R}^{2\times 2}$ denote the matrix with columns $\vec{b}_i$ and $\vec{b}_j$. Compute the matrix

$$C_{ij} := B_{ij}^{-1} A B_{ij}$$

for all three pairs $(i, j) \in \{(1,3), (1,4), (3,5)\}$.

$$\underbrace{\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}}_{C_{13}}, \quad \underbrace{\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}}_{C_{14}}, \quad \underbrace{\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}}_{C_{35}}.$$