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9. exercise sheet for Engineering Mathematics

9.1. *(Systems of linear differential equations)*
In this exercise, you should apply linear algebra to solve a system of linear differential equations. More precisely, the goal is to find two differentiable functions $x, y\colon \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} x(t) + 2y(t) \\ 2x(t) + y(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{†}$$

and

$$x(0) \stackrel{!}{=} 1, \quad y(0) \stackrel{!}{=} 3. \tag{‡}$$

(Here a dot above a function means the derivative with respect to t, that is, $\dot{x}(t) = x'(t)$.) The idea is to use eigenvalue theory in order to "decouple" the two dimensions inherent to the above problem, and pass to two one-dimensional problems, whose solution is much easier.

(a) Compute the eigenvalues and associated eigenvectors of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

(b) Find an invertible matrix $T \in \mathbb{R}^{2\times 2}$ such that $D = T^{-1}AT$ is a diagonal matrix.

(c) Find non-constant differentiable functions $u, v\colon \mathbb{R} \to \mathbb{R}$ such that, for all $t \in \mathbb{R}$,

$$\begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \end{pmatrix} \stackrel{!}{=} D \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$$

(Hint: try $t \mapsto \exp(\lambda t)$ and choose λ suitably.)

(d) Verify that $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} := T \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ satisfies (†).

(e) Replace your solutions u and v found in (c) with scalar multiples of themselves in such a way that the solution to (†) constructed in (d) also satisfies (‡).
(Hint: you can verify your solution using $(x(1), y(1)) \approx (39.803, 40.539)$.)

9.2. *(Diagonalising a symmetric matrix)*
Consider the symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 4 \\ -4 & 4 & -5 \end{pmatrix}.$$

(a) Find all eigenvalues of A. (Hint: consider the characteristic polynomial of A. One of its roots is -9. Use this to find all roots of the characteristic polynomial.)

(b) In (a) you should have seen that A has exactly two eigenvalues: -9 and some $\lambda \neq -9$. Find *two* eigenvectors $\vec{v}$, $\vec{w}$ for the eigenvalue λ and one eigenvector $\vec{z}$ for the eigenvalue -9 of A such that the matrix

$$T = \begin{pmatrix} | & | & | \\ \vec{v} & \vec{w} & \vec{z} \\ | & | & | \end{pmatrix}$$

is invertible. (Hint: in the present scenario, for the desired invertibility, it suffices to choose the $\vec{w}$ in such a way that it is not a scalar multiple of $\vec{v}$.)

(c) Compute $T^{-1}AT$. (Hint: the result should be a diagonal matrix.)

(d) Verify that $\vec{v} \cdot \vec{z} = 0$ and $\vec{w} \cdot \vec{z} = 0$, that is, $\vec{z}$ is perpendicular to $\vec{v}$ and $\vec{w}$. Moreover compute $\vec{v} \cdot \vec{w}$.

(e) If in (d), you got $\vec{v} \cdot \vec{w} \neq 0$, then find $\mu \in \mathbb{R}$ such that $\vec{w}' = \vec{w} - \mu\vec{v}$ satisfies $\vec{v} \cdot \vec{w}' = 0$. Verify that the matrix T' defined like T, but with the column $\vec{w}$ replaced by $\vec{w}'$ satisfies $(T')A(T')^{-1} = T^{-1}AT$. (Hint: getting $\vec{v} \cdot \vec{w} = 0$ in (d) right away is definitely possible and depends on your own choice of $\vec{v}$ and $\vec{w}$. If this happens to you, then you have nothing to do in (e).)

Remark: the above tasks are supposed to give a glimpse how one would diagonalise A using an orthogonal matrix (in the sense of exercise 7.3). The outcome of (e) (if solved correctly) is essentially such a matrix: $(T')^{\mathrm{T}}T'$ equals a diagonal matrix and by rescaling the columns of T one can arrange for this diagonal matrix to be equal to the 3×3-identity matrix $\mathbf{1}_3$.

9.3. *(Differentiation)*

Consider the two maps $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (xy^2, \exp(x))$, and $g\colon \mathbb{R}^2 \to \mathbb{R}$, $(v, w) \mapsto v - w$. Compute the following:

(a) $(g \circ f)(x, y)$;

(b) the Jacobian matrices $J_f(x, y)$, $J_g(v, w)$, and $J_{g\circ f}(x, y)$,

(c) the matrix–matrix product $J_g(f(x, y))J_f(x, y)$.

(Hint: examples for computing the Jacobian matrices can be found in § 5.1.3. You may verify your answer using $J_{g\circ f}(2, 3) \approx (1.61 \;\; 12) \in \mathbb{R}^{1\times 2}$.)