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10. exercise sheet for Engineering Mathematics

(first name) (last name)

(student id number)

10.1. *(Differentiation)* (4 credits)

Consider the two maps

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \cos y \\ x - y^4 \end{pmatrix}, \quad \text{and} \quad g\colon \mathbb{R}^2 \to \mathbb{R},\ \begin{pmatrix} v \\ w \end{pmatrix} \mapsto v + w^2.$$

Compute the following:

(a) $(g \circ f)(x, y)$; (Hint: for verification: $(g \circ f)(2,1) \approx 3.161$.)

$$(g \circ f)(x, y) = \boxed{}.$$

(b) the Jacobian matrices $J_f(x, y)$, $J_g(v, w)$, and $J_{g \circ f}(x, y)$,

$$J_f(x,y) = \begin{pmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{pmatrix}, \quad J_g(v,w) = \begin{pmatrix} \boxed{} & \boxed{} \end{pmatrix},$$

$$J_{g \circ f}(x,y) = \begin{pmatrix} \boxed{} & \boxed{} \end{pmatrix}.$$

(c) the matrix–matrix product $J_g(f(x, y))J_f(x, y)$. (Hint: at $(2,1)$: $\approx (4.161 \ \ {-11.366})$.)

$$\begin{pmatrix} \boxed{} & \boxed{} \end{pmatrix}.$$

Please submit your solutions during the next lecture (20.12.2023).
https://www.math.tugraz.at/~mtechnau/teaching/2023-w-engimaths.html

10.2. *(Gradient)* (4 credits)

Consider the map $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto \cos(x-y) - xy^2$.

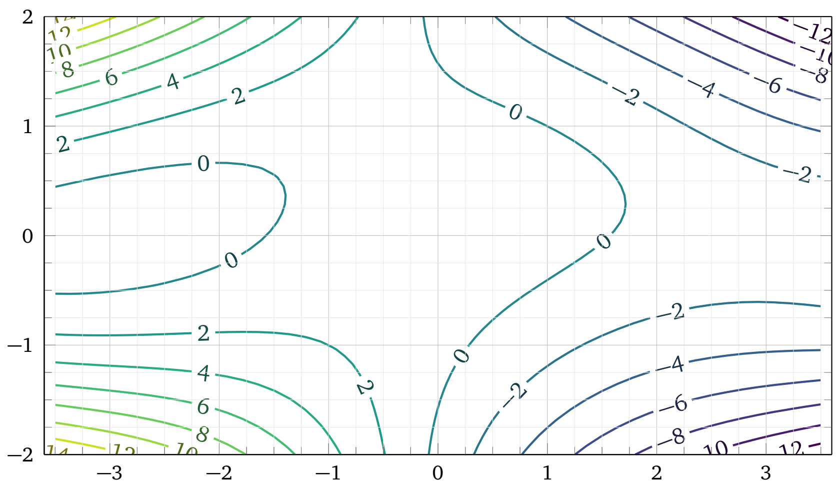
(a) Compute $J_f(x,y) \in \mathbb{R}^{1\times 2}$.

$$J_f(x,y) = \begin{pmatrix} \boxed{} & \boxed{} \end{pmatrix}.$$

(b) Compute $\operatorname{grad} f(x,y) \in \mathbb{R}^2$. (Hint: for verification: $\operatorname{grad} f(1,10) \approx (-99.588, -20.412)$.)

$$\operatorname{grad} f(x,y) = \begin{pmatrix} \boxed{} \\ \boxed{} \end{pmatrix}.$$

(c) Pick three distinct points $(x,y) \in [-3,3] \times [-2,2]$ for which you compute the gradient $\operatorname{grad} f(x,y)$ numerically and draw it as a vector based at (x,y) in the following picture:



(Hint: the curved lines are curves on which f is constant. For (c) make sure that the vectors you draw have the correct length and orientation.)

10.3. *(Divergence)* (4 credits)

Let $\vec{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (F_1(x,y), F_2(x,y))$ be a vector field. Define the ***divergence*** $\operatorname{div}\vec{F}(x,y)$ of $\vec{F}$ at (x,y) to be $\partial_1 F_1(x,y) + \partial_2 F_2(x,y)$ if the appearing partial derivatives exist. Compute $\operatorname{div}\operatorname{grad}(xy + y^2 - x)$.

(Hint: here the main task is to decipher the notation.)

$$\operatorname{div}\operatorname{grad} f(x,y) = \boxed{}.$$

10.4. *(Start-up of laminar flow in a tube, I)* (4 credits)

Let $\phi\colon [-1,1]\times\mathbb{R}_{\geq 0}\to\mathbb{R}$ be a (the) function satisfying the *partial differential equation*

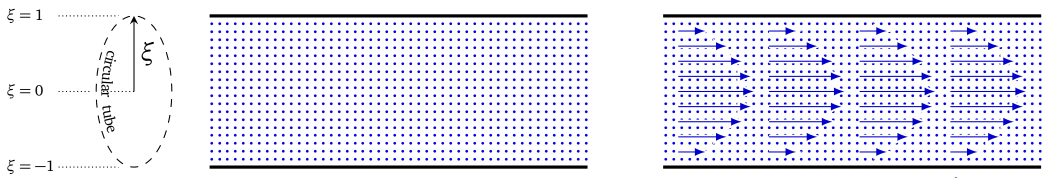
$$\frac{\partial\phi}{\partial\tau}(\xi,\tau) = 4 + \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi}{\partial\xi}(\xi,\tau)\right) \qquad (\partial)$$

at all points $(\xi,\tau)\in(-1,1)\times\mathbb{R}$, $\xi\neq 0$, and the *boundary conditions*

$$\phi(\pm 1,\tau) = 0 \quad \text{for } \tau\geq 0, \qquad (\dagger)$$

$$\phi(\xi,0) = 0 \quad \text{for } \xi\in[-1,1]. \qquad (\ddagger)$$

(Provided suitable physical assumptions which we do not explicate here, the function ϕ can be used to model the velocity profile of laminar flow building up from a fluid in a circular tube, initially at rest (imposed by ($\ddagger$)), with a pressure gradient applied at the ends of the tube at time $\tau = 0$. The boundary condition ($\dagger$) imposes zero velocity at the tube walls.[1])



Time $\tau = 0$ (fluid at rest), time $\tau > 0$ (arrows indicate fluid velocity; length $= \phi(\xi,\tau)$).

Let $y\colon\mathbb{R}\to\mathbb{R}$ be a solution to the Bessel differential equation studied in exercise 2.3. Let $\alpha > 0$ denote a number such that $y(\alpha) = 0$. Verify the following (on a separate sheet):

(a) ϕ_∞ given by $\phi_\infty(\xi,\tau) = 1-\xi^2$ satisfies[2]

- $\dfrac{\partial\phi_\infty}{\partial\tau}(\xi,\tau) = 0$,
- the partial differential equation (∂), and
- the boundary condition ($\dagger$), but not ($\ddagger$).

(b) $\phi_*\colon(\xi,\tau)\mapsto\phi_\infty(\xi)-\phi(\xi,\tau)$ satisfies the partial differential equation

$$\frac{\partial\phi_*}{\partial\tau}(\xi,\tau) = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi_*}{\partial\xi}(\xi,\tau)\right). \qquad (\partial')$$

(c) $\phi_\alpha\colon(\xi,\tau)\mapsto y(\alpha\xi)\exp(-\alpha^2\tau)$ satisfies (∂') and ($\dagger$).[3]

[1] For more context, consult Exercise 4D.2 of R. B. Bird, W. E. Stewart, E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: New York, 2001. In a later exercise, we shall use the above results in order to obtain a formula for ϕ. An animation of the outcome of this can be viewed here: `https://www.math.tugraz.at/~mtechnau/images/2023-w-engimaths-flow.gif`

[2] Physically, ϕ_∞ models the steady-state velocity profile that is approached by as time goes to infinity.

[3] Using a "separation *ansatz*", that is, looking for solutions to (∂') of the form $(\xi,\tau)\mapsto Y(\xi)E(\tau)$, one can actually deduce that Y must satisfy the Bessel differential equation from exercise 2.3 and E must satisfy $E' = cE$ for a suitable constant c related to Y. Here we are turning this approach on its head for simplicity's sake.