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11. exercise sheet for Engineering Mathematics

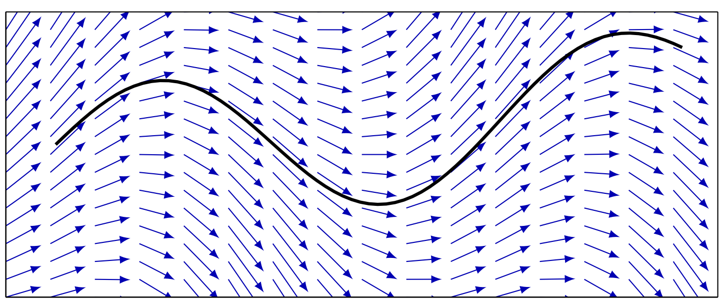
11.1. *(Substantial derivative)*

Suppose that $\vec{u}\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ models a time-dependent velocity field. That is, for any point $\vec{x} \in \mathbb{R}^3$ in space and any time $t \in \mathbb{R}$, the vector $\vec{u}(\vec{x}, t)$ is thought of as the velocity of some substance at the given point and time. Imagine an observer floating within the stream given by $\vec{u}$, described by their position function $\vec{X}\colon \mathbb{R} \to \mathbb{R}^3$ ($\vec{X}(t)$ is the position of the observer in space at time t). The position of the observer then satisfies

$$\dot{\vec{X}}(t) := J_{\vec{X}}(t) = \vec{u}(\vec{X}(t), t). \tag{†}$$

Suppose that to each point in space at any given time one also associates some scalar quantity of interest (e.g., the concentration of some substance dissolved in the fluid). This is modelled by yet another function $c\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. Imagine that the observer measures the value of c continuously at their current position at any given time.

The picture below illustrates the trajectory of the observer (black curve) in a time-independent velocity field (blue arrows). (The function c is not modelled in this picture.)



(a) Derive a formula for the rate of change of c *as observed by the observer*.
(Hint: you ought to differentiate $c(\vec{X}(t), t)$ [why?]. Use the chain rule, Theorem 5.6. In fluid dynamics, this is often written as $\frac{\mathrm{D}c}{\mathrm{D}t}$ and is called ***substantial derivative*** or ***material derivative***.)

(b) Now put $\vec{u}(\vec{x}, t) = (x_1, 0, 1)$, $\vec{X}(t) = (\exp(t), 5, t)$, and $c(\vec{x}, t) = x_3$.
Verify that $\vec{X}$ and $\vec{u}$ indeed satisfy (†).

(c) Compute $\dfrac{\mathrm{D}c}{\mathrm{D}t}$ for the choices from (b).

11.2. *(Potentials)*

(a) Find a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with

$$\operatorname{grad} f(x,y) = \begin{pmatrix} 2xy-1 \\ x^2 \end{pmatrix}.$$

(b) Find a function $g\colon \mathbb{R}^2 \to \mathbb{R}$ with

$$\operatorname{grad} g(x,y) = \begin{pmatrix} \sin(x-y) + x\cos(x-y) \\ -x\cos(x-y) \end{pmatrix}.$$

(Hint: expand the definition of the gradient and see what this tells you about the function f or g you need to find. Once you have f or g, it is also easy to check that your solution is correct; just compute the gradient.)

11.3. *(Taylor's formula)*

Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function which we shall assume to be infinitely often differentiable. In this exercise, we show the following (slightly simplified) version of Taylor's formula (6.3) from the lecture notes: for $n = 0, 1, 2, \ldots$ one has

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \int_0^x \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n \,\mathrm{d}\xi \quad \text{for all } x \in \mathbb{R}. \tag{A_n}$$

(a) Write out the statement (A_0) (that is, (A_n) with $n = 0$) and observe that the claimed equation holds for all $x \in \mathbb{R}$ due to the fundamental theorem of calculus.

(b) Show for $n = 0, 1, 2, \ldots$ that if (A_n) is true, then also (A_{n+1}) is true.
(Hint: start with the equation (A_n) and do an integration by parts, integrating the factor $(x-\xi)^n$, and differentiating $f^{n+1}(\xi)/n!$. For more hints consult § 6.2.2 of the lecture notes.)

Remark: (a) and (b) combine to show that (A_n) holds for every $n = 0, 1, 2, \ldots$; indeed, to see that, for instance, (A_5) holds, note that (A_0) holds by (a). Then (A_1) holds by (b). By applying (b) once more, we see that (A_2) holds, and applying (b) three more times, we infer that (A_5) holds. The procedure of proving the validity of some family of statements (B_0), (B_1), $\ldots$, parametrised by the (non-negative) integers by showing that the initial statement (B_0) holds and then showing that the truth of any statement (B_n) always implies the truth of the "subsequent" statement (B_{n+1}) is called ***mathematical induction***.

11.4. *(Evaluation)*

Until January 21, you have the opportunity to evaluate the present course and provide feedback via TUGRAZonline. Please consider doing so.