GRAZ UNIVERSITY OF TECHNOLOGY
INSTITUTE OF ANALYSIS AND NUMBER THEORY
Marc Technau

Winter term 2023
Graz, 10.01.2024

12. exercise sheet for Engineering Mathematics

______________________ ______________________
(first name) (last name)

☐☐☐☐☐☐☐☐
(student id number)

12.1. *(Taylor polynomials)* (4 credits)
For the functions f given below, compute their Taylor polynomials

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

of order $n = 0, 1, 2, 3$ at 0 ($x_0 = 0$ in the notation from § 6.2) and use a computer with software of your choice to plot the graphs of these polynomials along with the graph of f on the interval $(-1, 1)$.

(a) $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^5 + 2x^3 + x - 4$;

$$\sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k =$$

(b) $f\colon [-1, 1] \to [0, \pi]$, $x \mapsto \arccos(x)$.

$$\sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k =$$

(Hint: see § 6.2 in the lecture notes for more on Taylor polynomials. For the task of computing derivatives, you may want to take another look at § 0.6. It suffices to write down the Taylor polynomials for $n = 3$. For the plots, please use a separate sheet if you cannot superimpose them onto this sheet.)

Please submit your solutions during the next lecture (17.01.2024).
https://www.math.tugraz.at/~mtechnau/teaching/2023-w-engimaths.html

12.2. *(Newton's method)* (4 credits)

In this exercise, we shall use Newton's method to find numerical approximations to the roots of functions. (You can read up on Newton's method in § 6.3 of the lecture notes, but this exercise is self-contained.) Consider the function $\vec{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (x\exp(y) - 1, y - x - 1)$.

(a) Compute $J_{\vec{f}}(x,y) = \begin{pmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{pmatrix}$.

(b) Determine all $(x,y) \in \mathbb{R}^2$ for which the matrix $J_{\vec{f}}(x,y) \in \mathbb{R}^{2\times 2}$ is invertible and provide a formula for $J_{\vec{f}}(x,y)^{-1}$.

$$J_{\vec{f}}(x,y)^{-1} = \begin{pmatrix} \\ \\ \\ \end{pmatrix} \text{ for all } (x,y) \in \mathbb{R}^2 \text{ such that}\ldots$$

(c) Use your answer for (b) to find a formula for

$$\operatorname{Iter}(x,y) := (x,y) - J_{\vec{f}}(x,y)^{-1}\vec{f}(x,y),$$

assuming that (x,y) are such that $J_{\vec{f}}(x,y)$ is invertible.

$$\operatorname{Iter}(x,y) = \begin{pmatrix} \\ \\ \\ \end{pmatrix} \in \mathbb{R}^2.$$

(d) Set $\vec{x}_0 = (1,-1)$, $\vec{x}_1 = \operatorname{Iter}(\vec{x}_0)$, $\vec{x}_2 = \operatorname{Iter}(\vec{x}_1)$, $\vec{x}_3 = \operatorname{Iter}(\vec{x}_2)$. Use your formula from (c) and a calculator (or suitable software) to complete the following table:

i	0	1	2	3
$\vec{x}_i \approx$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} \square \\ \square \end{pmatrix}$	$\begin{pmatrix} \square \\ \square \end{pmatrix}$	$\begin{pmatrix} \square \\ \square \end{pmatrix}$
$\vec{f}(\vec{x}_i) \approx$	$\begin{pmatrix} \square \\ \square \end{pmatrix}$	$\begin{pmatrix} \square \\ \square \end{pmatrix}$	$\begin{pmatrix} \square \\ \square \end{pmatrix}$	$\begin{pmatrix} \square \\ \square \end{pmatrix}$

(Hint: here you are allowed [and encouraged] to use numerical approximations provided by your calculator. Otherwise you get iterated exponentials which quickly become very awkward. For $\vec{f}(\vec{x}_3)$ you should get a vector with quite small entries.)

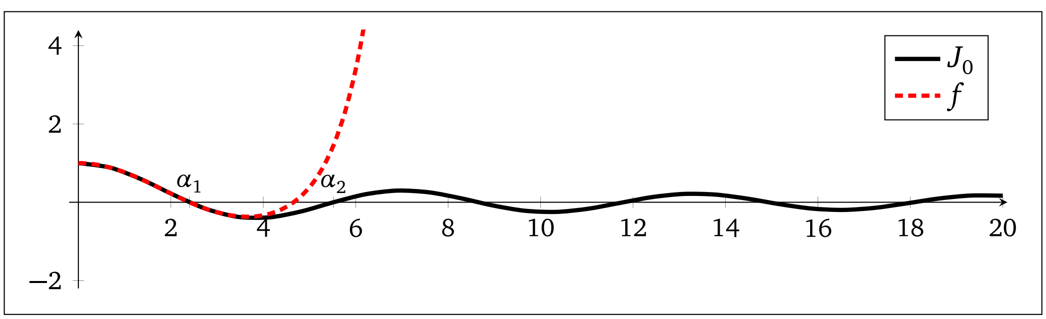
12.3. *(Start-up of laminar flow in a tube, II)* (4 credits)

Consider the Bessel function $J_0\colon \mathbb{R} \to \mathbb{R}$ of zeroth order and first kind, given by

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2} x^{2k}.$$

(You know this function from the solution of exercise 2.3, and you have plotted approximations to $4J_0$ there.) Here we shall be interested in finding approximations to the zeros of J_0. We approximate J_0 by its 8-th order Taylor polynomial about 0:

$$J_0(x) \approx \sum_{k=0}^{4} \frac{(-1)^k}{4^k (k!)^2} x^{2k} = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \frac{1}{147456}x^8 =: f(x).$$



(a) Compute two steps of Newton's method for f with initial value $x_0 = 2$ in order to get an approximation to a root of f. (Which, as we hope, does approximate a root of J_0 too.)

i	0	1	2
$x_i \approx$	2	$\square$	$\square$

(Hint: you can check your result for sanity by comparing x_2 to the list of roots given below.)

(b) Approximations to the first two roots of J_0 are given by

$$\alpha_1 = 2.404825558, \quad \alpha_2 = 5.520078112.$$

Using the formula

$$\sum_{\alpha} \frac{J_0(\alpha\xi)}{\alpha^3 J_0'(\alpha)} = -\frac{1}{8}(1-\xi^2),$$

(which we do not prove) where α ranges over all positive roots of J_0, one can show that the function ϕ from exercise 10.4 admits the following analytic representation:

$$\phi(\tau,\xi) = 1-\xi^2 + 8\sum_{\alpha} \frac{J_0(\alpha\xi)}{\alpha^3 J_0'(\alpha)} \exp(-\alpha^2\tau).$$

(Note that in the notation of exercise 10.4, ϕ is constructed as a suitable superposition of the functions ϕ_∞ and ϕ_α studied there.) We now approximate this via

$$\phi_{\approx}(\tau,\xi) = 1-\xi^2 + 8\frac{f(\alpha_1\xi)}{\alpha_1^3 f'(\alpha_1)} \exp(-\alpha_1^2\tau) + 8\frac{f(\alpha_2\xi)}{\alpha_2^3 f'(\alpha_2)} \exp(-\alpha_2^2\tau).$$

Use a computer to plot $\phi_{\approx}(\tau,\cdot)\colon [-1,1] \to \mathbb{R}$, $\xi \mapsto \phi_{\approx}(\tau,\xi)$, for the following three choices of τ: $\tau \in \{0, 1/5, 1\}$. (Hint: submit printouts of the plots. In exercise 10.4 you were given the link to an animation showing $\phi(\tau,\cdot)$, as τ increases. If your computations are correct, then you should notice obvious similarities between your plots and the animation.)