GRAZ UNIVERSITY OF TECHNOLOGY
INSTITUTE OF ANALYSIS AND NUMBER THEORY
Marc Technau

Winter term 2023
Graz, 17.01.2024

13. exercise sheet for Engineering Mathematics

13.1. *(Polar coordinates, differentiation)*
Consider the function

$$f\colon \mathbb{R}^2 \setminus \{\vec{0}\} \to \mathbb{R}, \quad (x,y) \mapsto \frac{2xy}{(x^2+y^2)^3},$$

as well as the well-known polar coordinate map $\vec{P}\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(r,\varphi) \mapsto (r\cos\varphi, r\sin\varphi)$. Let $\vec{v} = (1/\sqrt{2}, 1/\sqrt{2})$. Compute the following quantities.

(a) $\partial_1 f(x,y)$ and $\partial_2 f(x,y)$.

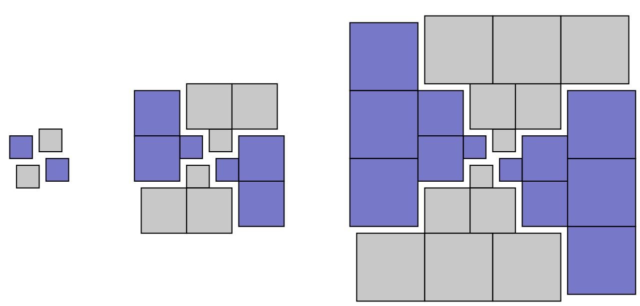
(b) $\dfrac{\partial f}{\partial \vec{v}}(x,y)$.

(c) $(f \circ \vec{P})(r,\varphi)$.

(d) $\dfrac{\partial f}{\partial r}(r,\varphi)$ and $\dfrac{\partial f}{\partial \varphi}(r,\varphi)$.
(Hint: this notation means $\partial_1(f \circ \vec{P})$ and $\partial_2(f \circ \vec{P})$; see Example 6.6.)

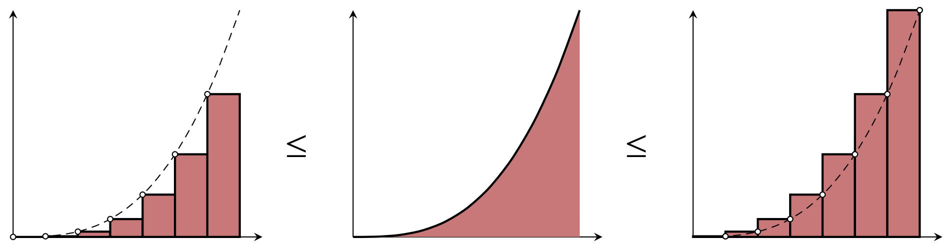
13.2. *(Sums and integrals)*
This exercise is aimed at getting a feel for the definition of the Darboux integral (see Chapter 7 of the lecture notes).

(a) Stare at the picture below and use this to find a formula for $\sum_{k=1}^{n} k^3 = 1^3+2^3+\ldots+n^3$.



(b) Using a partition of $[0,1]$ into n pieces of equal length, the definition of the Darboux integral yields

$$\sum_{k=1}^{n}\left(\frac{k-1}{n}\right)^3 \frac{1}{n} \le \int_0^1 x^3\,\mathrm{d}x \le \sum_{k=1}^{n}\left(\frac{k}{n}\right)^3 \frac{1}{n}.$$



Use your answer from (a) to compute the outer two sums. Also compute the integral in the middle and show that, as $n \to \infty$, both sums converge to that integral (this corresponds to taking the partition $0, 1/n, 2/n, \dots, 1$ of $[0,1]$ to be much finer).

13.3. *(A variant of the Gaussian integral)*

Compute $\displaystyle\int_{-\infty}^{\infty} \exp(-x^2 + x)\,\mathrm{d}x$.

(Hint: call the "completing the square" trick from your school days and use the result from Example 7.3.)

13.4. *(Volume of a solid)*

Compute the volume of

$$R = \{\, (x, y, z) \in \mathbb{R}^3 : 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le xy \,\}.$$

(Hint: compute $\displaystyle\int_R 1\,\mathrm{d}^3\vec{x}$ using Fubini's theorem. A rough sketch of R is given below.)