GRAZ UNIVERSITY OF TECHNOLOGY
INSTITUTE OF ANALYSIS AND NUMBER THEORY
Marc Technau

Winter term 2023
Graz, 24.01.2024

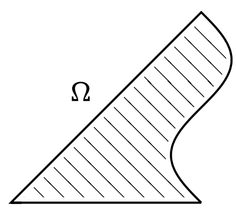
14. exercise sheet for Engineering Mathematics

(first name) (last name)

(student id number)

14.1. *(Area computation)* (4 credits)
Compute the area of $\Omega = \{ (x,y) \in \mathbb{R}^3 : 0 \le x \le 2\pi - \sin(y),\ 0 \le y \le x \}$.

$$\text{area}(\Omega) = \boxed{}.$$



(Hint: see the solution of exercise 13.4. The answer is approximately 19.7.)

14.2. *(Volume of a solid, II)* (4 credits)
Compute the volume of $R = \{ (x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le xy \}$.

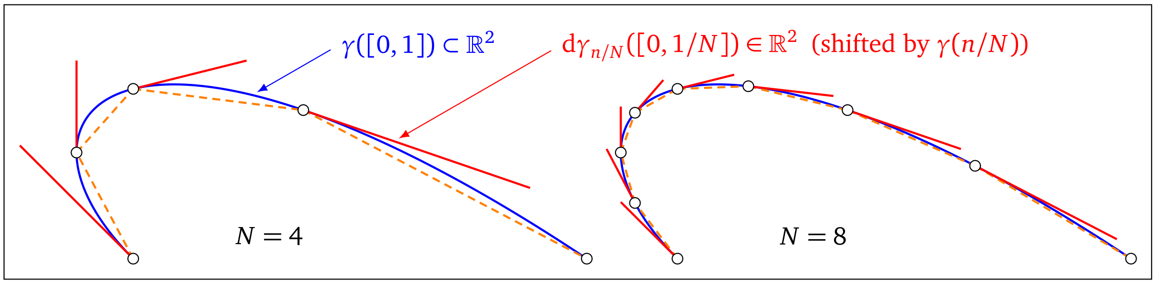
$$\text{vol}(R) = \boxed{}.$$

(Hint: see the solution of exercise 13.4. The answer is roughly 0.04.)

Please submit your solutions during the next lecture (31.01.2024).
https://www.math.tugraz.at/~mtechnau/teaching/2023-w-engimaths.html

14.3. *(Length of a curve)* (4 credits)

Consider the image $\gamma([0,1]) = \{\gamma(t) : t \in [0,1]\}$ of $[0,1]$ under the function $\gamma\colon \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (2t^2 - t, t - t^3)$. It is a curve in $\mathbb{R}^2$:



Let $\tau > 0$ and put $N = 4$. Compute the following:

(a) $\mathrm{d}\gamma_t\colon \mathbb{R} \to \mathbb{R}^2$;

$$\mathrm{d}\gamma_t(\tau) = \left(\boxed{}, \boxed{} \right) \in \mathbb{R}^2.$$

(b) the length of $\mathrm{d}\gamma_t([0,\tau]) =$ $\boxed{}$;

(c) $\sum_{n=0}^{N-1} \|\gamma((n+1)/N) - \gamma(n/N)\| \approx$ $\boxed{}$;

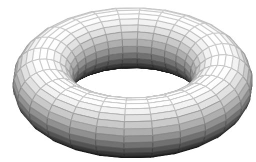
(d) $\sum_{n=0}^{N-1} \mathrm{length}(\mathrm{d}\gamma_{n/N}([0,1/N])) \approx$ $\boxed{}$.

(Hints: for (b), see exercise 6.3; for $\tau = t = 2$ your answer should be ≈ 26. For (c) and (d), numerical approximations suffice; you may use a calculator for obtaining these. The answers are roughly 1.5 and 1.3, but your answers should have higher accuracy.)

14.4. *(Area of a torus)* (4 credits)

Let $R > r > 0$ and put $U = [0, 2\pi)^2$. The map

$$\vec{\Phi}\colon U \to \mathbb{R}^3, \qquad \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} R\cos(u) + r\cos(u)\cos(v) \\ R\sin(u) + r\sin(u)\cos(v) \\ r\sin(v) \end{pmatrix},$$



parametrises a torus $T = \vec{\Phi}(U)$ (also known as a "doughnut") with outer radius R and inner radius r. Compute the following:

(a) $$J_{\vec{\Phi}}(u,v) = \begin{pmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{pmatrix}.$$

(b) $$\det(J_{\vec{\Phi}}(u,v)^{\mathrm{T}} J_{\vec{\Phi}}(u,v)) = \boxed{}.$$

(Hint: the matrix whose determinant is to be computed should turn out to be a diagonal matrix.)

(c) $$\operatorname{area}(T) = \iint_T 1 \, \mathrm{d}A = \int_U \sqrt{\det(J_{\vec{\Phi}}(u,v)^{\mathrm{T}} J_{\vec{\Phi}}(u,v))} \, \mathrm{d}^2(u,v) = \boxed{}.$$

(Hint: you may verify your result by checking that, for $r = 1$ and $R = 3$, your formula yields ≈ 118.435.)

(Remark: to know that in (c) you are really computing the *area of T*, you should read § 7.2 of the lecture notes [which has not yet been discussed in the lecture], but the exercise is phrased in such a way that you only require Fubini and your knowledge from the previous chapters.)