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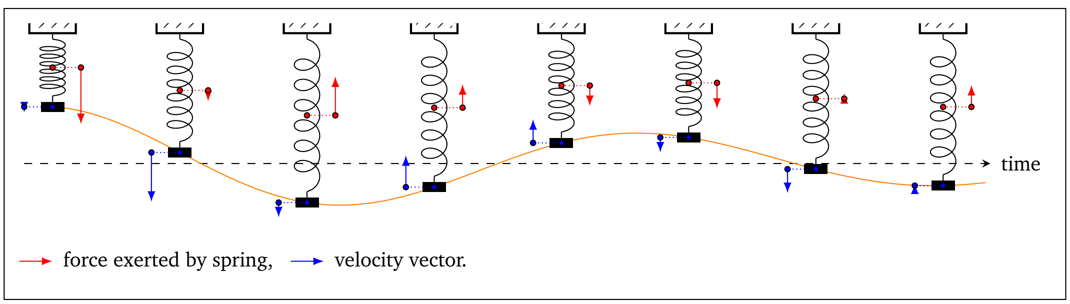
1. exercise sheet for Mathematics for Advanced Materials Science

1.1. *(Linear ordinary differential equations)*
Consider the differential equation

$$\ddot{x} + \dot{x} + x \overset{!}{=} 0. \qquad (\dagger)$$

(This equation may be viewed as modelling the evolution of the displacement of a mass on a spring with friction taken into account. To determine the behaviour exactly, one needs to further conditions, such as the initial position of velocity of the mass at a given starting time.)



(a) Proceed as in § 1.1 of the lecture notes in order to find λ such that $x_\lambda(t) = \exp(\lambda t)$ solves ($\dagger$).
(Hint: you will have to find complex roots of a quadratic polynomial; from school you know a formula for this. Use $\sqrt{-1} = \pm\mathrm{i}$.)

(b) As in the lecture, you can multiply the solutions found in (a) by constants and also add two solutions, and still get a solution to ($\dagger$). Use this to find a function $x : \mathbb{R} \to \mathbb{R}$ solving the differential equation ($\dagger$) and satisfying $x(0) = 0$ and $\dot{x}(0) = 1$.
(Hint: look up Theorem 1.3 in § 1.5 of the lecture notes and use the formulas given there to find solutions involving the exponential function, the sine and cosine function, but non-real complex numbers. You may test your solution for correctness by verifying, using a calculator, that $x(1) \approx 0.533507$.)

(c) Find *two different* solutions $x : \mathbb{R} \to \mathbb{R}$ to the differential equation ($\dagger$) with $x(0) = 1$.

1.2. *(Computing with complex numbers)*
For this exercise, please read the rest of § 1.2 of the lecture notes (about one page) and note especially the examples at the end of that section. Write the following complex numbers in the form $a + \mathrm{i}b$ with real numbers a and b.

(a) $\dfrac{1}{2+\mathrm{i}}$,

(b) $\dfrac{2+4\mathrm{i}}{1+3\mathrm{i}} - 2 + \mathrm{i}$,

(c) $\left|\dfrac{1}{2+\mathrm{i}}\right|$,

(d) $\sqrt{3+4\mathrm{i}}$.

(Hint for (d): find two complex numbers $z = a + \mathrm{i}b$ with $z^2 = 3 + 4\mathrm{i}$.)

1.3. *(Equation for a circle)*
Let r be a positive real number and w a complex number. The set of $z \in \mathbb{C}$ satisfying $|z - w| = r$ is a circle with centre w and radius r. Show that $|z - w| = r$ holds if and only if $z\overline{z} - w\overline{z} - \overline{w}z + (w\overline{w} - r^2) = 0$.
(Hint: one can extend this further to derive equations capable of describing both circles and lines, which is mathematically quite pleasing and can simplify certain computations in geometry. Here, however, this is just meant to train computing with complex numbers. In particular, you should recall how the absolute value of a complex number relates to the complex conjugate of that number. Then you should compute.)