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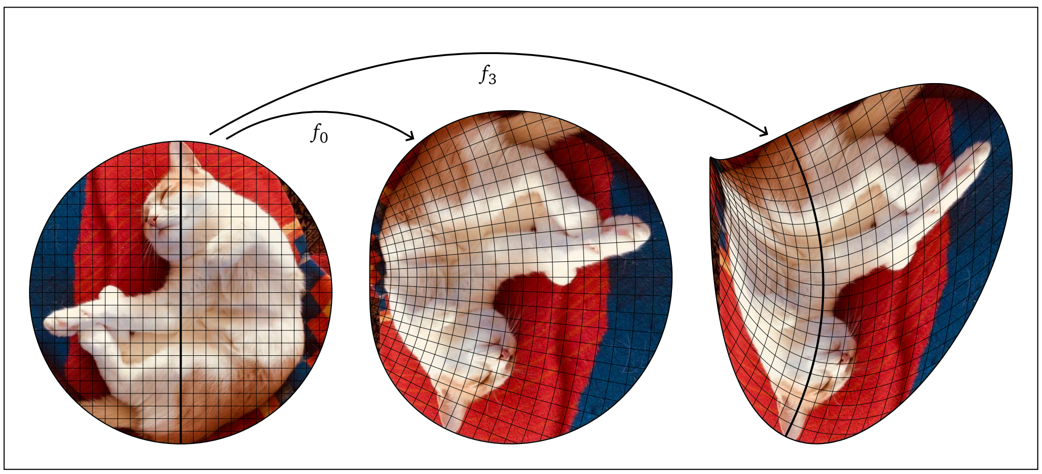
3. exercise sheet for Mathematics for Advanced Materials Science

3.1. *(Complex differentiation)*
Consider the map $f_U : \mathbb{C} \to \mathbb{C}$, $x + \mathrm{i}y \mapsto (4 - 4x + x^2 - y^2) + \mathrm{i}(Ux^2 - 4y + 2xy)$.



(a) Determine all points $z = x + \mathrm{i}y \in \mathbb{C}$ at which f_0 is complex-differentiable.

(b) Determine all points $z = x + \mathrm{i}y \in \mathbb{C}$ at which f_3 is complex-differentiable.

(Hints: for (a) compute $(x + \mathrm{i}y + r)^2$. For (b) study complex-differentiability of $f_3 - f_0$ first; for this check out the argument in the lecture notes that complex conjugation is not complex-differentiable. Then use this and (a) to deduce an answer for (b).)

3.2. *(Complex and real forms of Fourier series)*
Let $c_0, c_{\pm 1}, \ldots, c_{\pm K}$ be complex numbers. Find complex numbers a_k and b_k such that for every real x

$$\sum_{k=-K}^{K} c_k \exp(\mathrm{i}kx) = \frac{a_0}{2} + \sum_{k=1}^{K} (a_k \cos(kx) + b_k \sin(kx)).$$

(Remark: when sending $K \to \infty$, the above expressions turn into what is called "Fourier series". We shall treat these in the lecture in due time. In the literature both forms shown in the above equations do appear. Therefore, it is useful to know how to pass from one side to the other.)

3.3. *(Computing with the complex exponential function)*
For real x, write the following complex number in the form $a + \mathrm{i}b$ with real numbers a

and b.

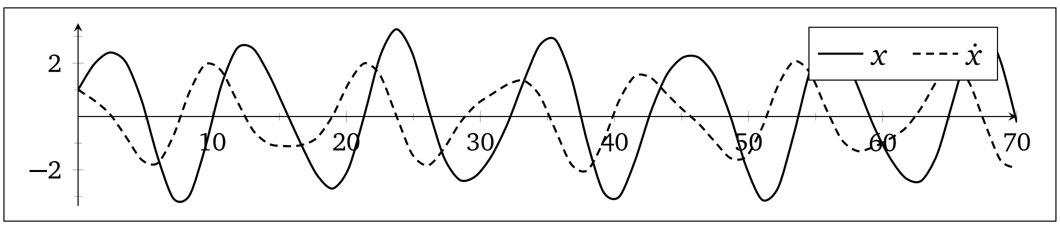
$$\sum_{\substack{k=-3\\k\neq 0}}^{3} \frac{\mathrm{i}}{k} \exp(2\pi \mathrm{i} k x).$$

(Hint: the sum is over $k = \pm 1, \pm 2, \pm 3$ without $k = 0$. You should get some sines or cosines depending on x; the imaginary part b should look particularly simple.)

3.4. *(Laplace transform)*
Let x be a solution to the following initial value problem:

$$\begin{cases} \text{differential equation:} & 3\ddot{x} + x \overset{!}{=} \sin \text{ on } \mathbb{R}_+, \\ \text{initial conditions:} & \begin{cases} \dot{x}(0) \overset{!}{=} 1, \\ x(0) \overset{!}{=} 1. \end{cases} \end{cases}$$

Find the Laplace transform $\mathscr{L}\{x\}$ of x.



(Hint: to solve this exercise, you just need the three properties of $\mathscr{L}\{\cdot\}$ presented in the lecture and the fact that $\mathscr{L}\{\sin\}(s) = 1/(s^2+1)$, which you may use without proof. You may check your solution using $\mathscr{L}\{x\}(0) = 4$ and $\mathscr{L}\{x\}(2) \approx 0.70769$. For a worked example, consult § 2.1.1 of the lecture notes.)