GRAZ UNIVERSITY OF TECHNOLOGY
INSTITUTE OF ANALYSIS AND NUMBER THEORY
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4. exercise sheet for Mathematics for Advanced Materials Science

(first name)	(last name)
(student id number)	

4.1. *(Laplace transform)* (4 credits)
Find $\mathscr{L}\{f\}$ where $f(t) = t\sin(t)\exp(t)$.

$$\mathscr{L}\{f\}(s) = \boxed{}.$$

(Hint: use $\mathscr{L}\{f\}(4) = 0.06$ to verify your final result. To find the solution you can try to use integration by parts a couple of times. If done correctly, integrating by parts four times should suffice. Alternatively, you are free to use Proposition 2.4 and Table 1 from the lecture notes. The latter option is probably the easier of the two.)

4.2. *(Laplace transform)* (4 credits)
In exercise 3.4 you have computed

$$\mathscr{L}\{x\}(s) = \frac{3s(s^2+s+1)+4}{3s^4+4s^2+1} \tag{†}$$

for the solution x to the following initial value problem:

$$\begin{cases} \text{differential equation:} & 3\ddot{x} + x \overset{!}{=} \sin \ \text{ on } \mathbb{R}_+, \\ \text{initial conditions:} & \begin{cases} \dot{x}(0) \overset{!}{=} 1, \\ x(0) \overset{!}{=} 1. \end{cases} \end{cases}$$

Please submit your solutions during the next lecture (16.11.2023).
https://www.math.tugraz.at/~mtechnau/teaching/2023-w-mams.html

The goal is to invert the Laplace transform in order to find an expression for x. To this end, compute the partial fraction decomposition of (†). In this particular setting, find eight complex numbers to put in the following boxes in order to make the equation work:

$$\mathscr{L}\{x\}(s) = \frac{\boxed{}}{s - \left(\boxed{}\right)} + \frac{\boxed{}}{s - \left(\boxed{}\right)} + \frac{\boxed{}}{s - \left(\boxed{}\right)} + \frac{\boxed{}}{s - \left(\boxed{}\right)}.$$

Use this to find an expression for x.

$$x(t) = \boxed{}.$$

(Hint: you can use $x(1) \approx 1.8352$ and $x(2) \approx 2.3259$ to verify your result. An explanation of partial fraction decomposition and how to compute it, can be found in § 2.5 of the lecture notes.)

4.3. *(Laplace transform)* (4 credits)
Find a function f with $\mathscr{L}\{f\}(s) = \dfrac{s-2}{s^2+4}$.

$$f(t) = \boxed{}.$$

(Hint: you can use $f(1) \approx 0.10316$ and $f(\pi) = 1$ to verify your result.)

4.4. *(Laplace transform)* (4 credits)
Compute

$$\mathscr{L}\{t \mapsto e^{it}\}(s) = \int_0^\infty e^{it} e^{-st}\,\mathrm{d}t$$

and *use this* to deduce the following formulae:

(a) $\mathscr{L}\{\cos\}(s) = \dfrac{s}{s^2+1}$, and

(b) $\mathscr{L}\{\sin\}(s) = \dfrac{1}{s^2+1}$.

(Remark: unlike the other exercises above, this one actually asks for the computation that takes you to the final result. If you run out of space here, please use a separate sheet. The point of the present exercise is two-fold: (1) to see that one can also compute the Laplace transform "by hand" from the definition, and (2) to see once more that complex numbers can on occasion help to simplify computations.)