GRAZ UNIVERSITY OF TECHNOLOGY
INSTITUTE OF ANALYSIS AND NUMBER THEORY
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Winter term 2023
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6. exercise sheet for Mathematics for Advanced Materials Science

____________________ (first name) ____________________ (last name)

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6.1. *(Inverting matrices)* (4 credits)

Find the inverse matrix A^{-1} of

$$A = \begin{pmatrix} 2 & -3 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad A^{-1} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}.$$

(Hint: there are several ways of doing this. For example, you may use the Gauß–Jordan algorithm from § 3.6.5 of the lecture notes. Alternatively, you may use Cramer's rule, Proposition 3.2. The final result will actually have all integer entries. You may check your answer by computing the matrix–matrix product $A^{-1}A$ and verifying that it equals the 3×3 identity matrix $\mathbf{1}_3$.)

6.2. *(Finding certain linear maps)* (4 credits)

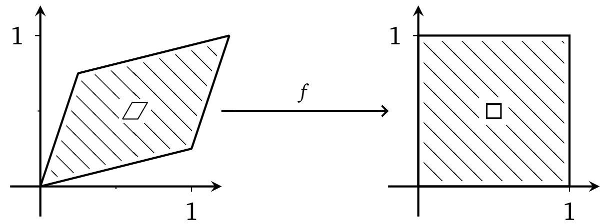
Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that the associated linear map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{v} \mapsto A\vec{v}$, maps the parallelogram

$$▱ = \{\, (x,y) \in \mathbb{R}^2 : 0 \le \tfrac{12}{11}x - \tfrac{4}{11}y \le 1,\ 0 \le \tfrac{16}{11}y - \tfrac{4}{11}x \le 1 \,\}$$

Please submit your solutions during the next lecture (30.11.2023).
`https://www.math.tugraz.at/~mtechnau/teaching/2023-w-mams.html`

onto the unit square $\square = [0,1]\times[0,1]$, i.e., $f(\varparallelogram) := \{ f(\vec{v}) : \vec{v} \in \varparallelogram \} = \square$:

$$A = \begin{pmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{pmatrix}.$$



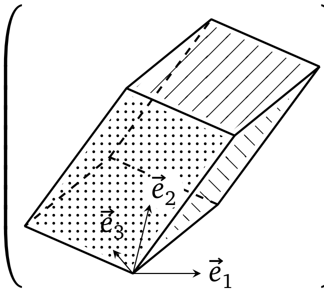
(Hint: it may be easier to find a matrix $B \in \mathbb{R}^{2\times 2}$ such that the associated linear map maps $\square$ onto $\varparallelogram$. One may then take $A = B^{-1}$. You can check your result easily on your own, by verifying that it maps the vertices of the parallelogram to the vertices of the square.)

6.3. *(Volume of a parallelepiped)* (4 credits)
Compute the volume of the parallelepiped

$$\varparallelogram := \varparallelogram(\vec{v},\vec{w},\vec{z}) := \{ \lambda_1\vec{v} + \lambda_2\vec{w} + \lambda_3\vec{z} : 0 \le \lambda_1,\lambda_2,\lambda_3 \le 1 \}.$$

spanned by the vectors $\vec{v} = (1,1,0)$, $\vec{w} = (1,2,0)$ and $\vec{z} = (-1,0,2)$.

$$\text{volume}(\varparallelogram) = \text{volume}\left(\quad \right) = \boxed{}.$$



(Hint: you can use Cavalieri's principle, or you can simply compute an appropriate determinant. For more details, see § 3.2 in the lecture notes. The final answer should be close to 2.)

6.4. *(Computing determinants)* (4 credits)
Let $r, \varphi, \theta \in \mathbb{R}$. Compute:

(a) $\det\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix} = \boxed{}$,

(b) $\det\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \boxed{}$,

(c) $\det\begin{pmatrix} \cos(\varphi)\sin(\theta) & r\cos(\varphi)\cos(\theta) & -r\sin(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) & r\sin(\varphi)\cos(\theta) & r\cos(\varphi)\sin(\theta) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{pmatrix} = \boxed{}$.

(Hint: see § 3.2 in the lecture notes. For (c), employ the identity $\cos(\varphi)^2 + \sin(\varphi)^2 = |\exp(i\varphi)| = 1$ from Theorem 1.3. Your final result should only depend on r and θ and look very simple.)