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7. exercise sheet for Mathematics for Advanced Materials Science

7.1. *(Solving systems of linear equations with a parameter)*

For $x \in \mathbb{R}$, consider the matrix $A_x = \begin{pmatrix} x-1 & 2 \\ 2 & x-1 \end{pmatrix} \in \mathbb{R}^{2\times 2}$.

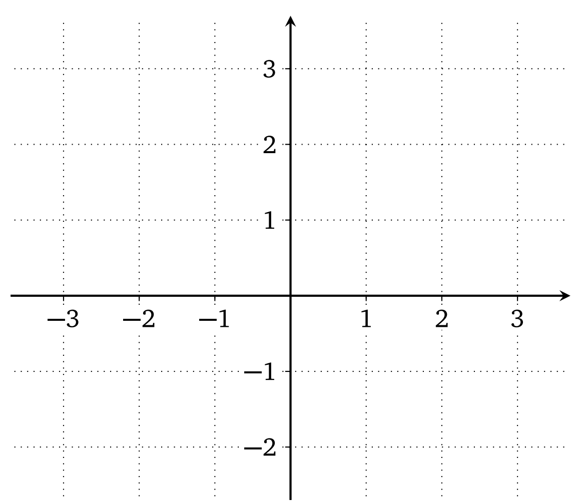
(a) Find *all* values of x such that the system of linear equations given by $A_x \vec{v} \overset{!}{=} \binom{0}{0}$ admits a solution $\vec{v} \in \mathbb{R}^2$ different from the zero vector. (Hint: one can deduce from Cramer's rule that it suffices to consider the x such that $\det A_x = 0$.)

(b) For each x determined above, provide a non-zero solution $\vec{v}$ to the above system.

7.2. *(Gram determinants)*

Consider the matrix $A = \binom{1}{3} \in \mathbb{R}^{2\times 1}$ and the associated linear map $f\colon \mathbb{R}^1 \to \mathbb{R}^2$, $v \mapsto Av$.

(a) Sketch the image $\operatorname{im} f = \{ f(v) : v \in \mathbb{R} \} \subseteq \mathbb{R}^2$ of f below:



(b) In your above sketch, mark the part of $\operatorname{im} f$ that is $\{ f(v) : 0 \le v \le 1 \}$ and determine its length.

(c) Compute $\sqrt{\det(A^{\mathrm{T}}A)}$ and $\sqrt{\det(AA^{\mathrm{T}})}$.

7.3. *(Area of a triangle)*

Compute the area of the two triangles with the following vertices:

(a) $(0,0)$, $(1,2)$ and $(1,3)$ in $\mathbb{R}^3$.

(b) $(0,0,0)$, $(1,2,3)$ and $(1,3,3)$ in $\mathbb{R}^3$.

(c) $(0,0,0,0,0,0,0)$, $(1,1,0,2,1,1,1)$ and $(1,2,1,0,1,0,1)$ in $\mathbb{R}^7$.

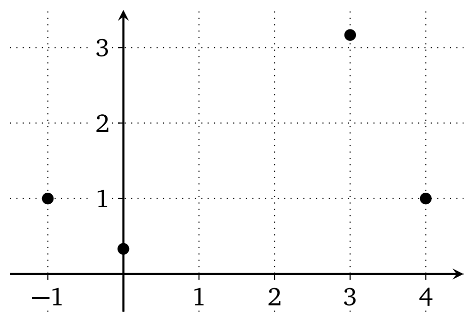
(Hint: ◢◤. The answers are roughly 0.5, 1.6 and 3.4 respectively.)

7.4. *(Linear regression)*

Consider the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \in \mathbb{R}^{4\times 2}$ and the vector $\vec{b} = \begin{pmatrix} 1 \\ 1/3 \\ 19/6 \\ 1 \end{pmatrix} \in \mathbb{R}^4$.

(a) Solve the system of linear equations $A^{\mathrm{T}}A\vec{x} \stackrel{!}{=} A^{\mathrm{T}}b$ for $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$.

(b) With your solution $\vec{x}$ from above, sketch the graph of the affine map $f\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto x_1 + x_2 t$, below:



(The black points are $(-1,1)$, $(0,1/3)$, $(3,19/6)$ and $(4,1)$.)

(c) Using the function f from the previous exercise, compute

$$\mathscr{E}_f := (1 - f(-1))^2 + (1/3 - f(0))^2 + (19/6 - f(3))^2 + (1 - f(4))^2. \qquad (\star)$$

(Hint: the final solution may look slightly ugly, but it is roughly 3.5.)

(d) Pick a vector $(y_1, y_2) \in \mathbb{R}^2$ other than $\vec{x}$ and compute the quantity in $(\star)$ with f replaced by $g\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto y_1 + y_2 t$. Also sketch the graph of g in the figure in (b).

(Remark: you may consult § 3.2.6 of the lecture notes for some general context on this exercise.)