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# 9. exercise sheet for Mathematics for Advanced Materials Science

**9.1.** *(Eigenvalues and eigenvectors, I)*
Consider $(C, n) \in \{(A, 2), (B, 3)\}$, where $A$ and $B$ are the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}.$$

For *both* choices of $(C, n)$ do the following:

(a) determine the characteristic polynomial $\chi_C = \det(X\mathbf{1}_n - C)$ (here "$X$" should be treated like a variable; think of your favourite number, but do not plug it in),

(b) compute the eigenvalues of $C$ (= the numbers $\lambda$ that yield zero when substituted for $X$ in the polynomial $\chi_C$) and all associated eigenvectors (= the non-zero solutions $\vec{v} \in \mathbb{R}^n$ of $(\lambda\mathbf{1}_n - C)\vec{v} \overset{!}{=} \vec{0}$),

(c) and discern whether the matrix $C$ is diagonalisable or not (i.e., decide whether you can choose eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ such that the matrix with these eigenvectors as columns has non-zero determinant).

(Hint: you can find some worked examples in § 3.5 of the lecture notes.)

**9.2.** *(Eigenvalues and eigenvectors, II)*
Consider the matrix $A \in \mathbb{R}^{2\times 2}$ and the vectors $\vec{b}_1, \ldots, \vec{b}_5 \in \mathbb{R}^2$ given below:

$$A = \begin{pmatrix} 11 & -12 \\ 8 & -9 \end{pmatrix}, \quad \vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \vec{b}_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{b}_4 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \vec{b}_5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(a) For each vector $\vec{b}_j$ ($j = 1, \ldots, 5$), check whether it is an eigenvector of $A$ and, if it is, determine the corresponding eigenvalue.

(b) Let $B_{ij} \in \mathbb{R}^{2\times 2}$ denote the matrix with columns $\vec{b}_i$ and $\vec{b}_j$. Compute the matrix

$$C_{ij} := B_{ij}^{-1} A B_{ij}$$

for all three pairs $(i, j) \in \{(1, 3), (1, 4), (3, 5)\}$.

**9.3.** *(Systems of linear differential equations)*
In this exercise, you should apply linear algebra to solve a system of linear differential

equations. More precisely, the goal is to find two differentiable functions $x, y\colon \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} x(t) + 2y(t) \\ 2x(t) + \phantom{2}y(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{†}$$

and

$$x(0) \stackrel{!}{=} 1, \quad y(0) \stackrel{!}{=} 3. \tag{‡}$$

(Here a dot above a function means the derivative with respect to $t$, that is, $\dot{x}(t) = x'(t)$.)

(a) Compute the eigenvalues and associated eigenvectors of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

(b) Find an invertible matrix $T \in \mathbb{R}^{2\times 2}$ such that $D = T^{-1}AT$ is a diagonal matrix.

(c) Find differentiable functions $u, v\colon \mathbb{R} \to \mathbb{R}$ such that, for all $t \in \mathbb{R}$,

$$\begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \end{pmatrix} \stackrel{!}{=} D \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$$

(Hint: try $t \mapsto \exp(\lambda t)$ for suitable $\lambda$.)

(d) Verify that $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} := T \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ satisfies (†).

(e) Replace your solutions $u$ and $v$ found in (c) with scalar multiples of themselves in such a way that the solution to (†) constructed in (d) also satisfies (‡).

**9.4.** *(Fourier series, I)*
Consider the 1-periodic function $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto 5 + 2\cos(2\pi x)$.

(a) Compute the Fourier coefficients

$$\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x} \, dx.$$

of $g$ for $k \in \mathbb{Z}$.
(Hint: you should get $\hat{f}(0) = 5$, $\hat{f}(\pm 1) \neq 0$, and $\hat{f}(k) = 0$ for all other $k$.)

(b) Determine at which points $f$ is represented by its Fourier series, i.e., for which $x \in \mathbb{R}$ does

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x} \,?$$

(Hint: you can use Theorem 4.1, but in the present case, the infinite series will turn out to be a finite sum, because almost all $\hat{f}(k)$ are zero, as you will know from (a). In this very special case, one can also argue using Theorem 1.3.)