Graz University of Technology
Institute of Analysis and Number Theory
Marc Technau

Winter term 2023
Graz, 11.01.2024

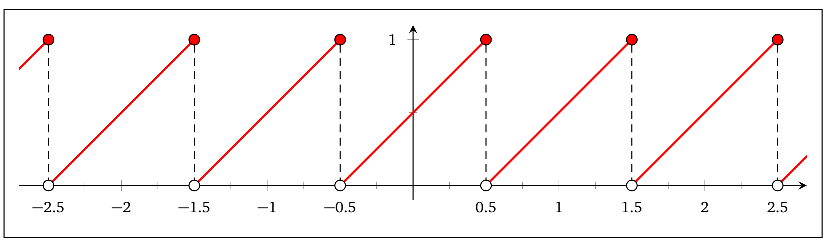
10. exercise sheet for Mathematics for Advanced Materials Science

(first name) (last name)

(student id number)

10.1. *(Fourier series, I)* (4 credits)

Let $g\colon \mathbb{R} \to \mathbb{R}$ be the 1-periodic function defined by $g(x) = x + 1/2$ for $|x| < 1/2$ and $g(1/2) = 1$. (In particular, $g(-1/2) = g(-1/2+1) = g(1/2) = 1$.)



(a) Compute the Fourier coefficients $\hat{g}(k)$ of g for $k \in \mathbb{Z}$. (Hint: $\hat{g}(1) \approx -0.159\mathrm{i}$, $\hat{g}(8) \approx 0.019894\mathrm{i}$. The solution can *almost* be found in § 4.3 of the lecture notes.)

$\hat{g}(0) =$ ☐ and $\hat{g}(k) =$ ☐ (for $k \neq 0$).

(b) Determine at which points g is represented by its Fourier series, i.e., for which $x \in \mathbb{R}$ does

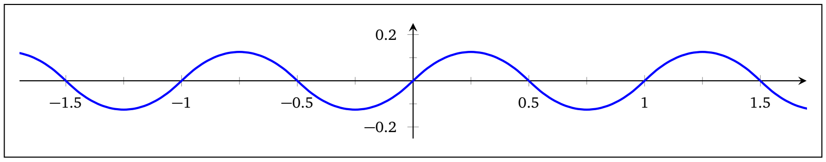
$$g(x) = \sum_{k=-\infty}^{\infty} \hat{g}(k) e^{2\pi \mathrm{i} k x}\ ? \quad \text{Answer: all } x \in \square .$$

Please submit your solutions during the next lecture (18.01.2024).

https://www.math.tugraz.at/~mtechnau/teaching/2023-w-mams.html

10.2. *(Fourier series, II)* (4 credits)

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the 1-periodic function defined by $f(x) = (1-2|x|)x$ for $|x| \leq 1/2$.



(a) Compute the Fourier coefficients $\hat{f}(k)$ of f for $k \in \mathbb{Z}$. (Hint: this is an exercise in partial integration and requires a bit of tenacity. You may use the following values to verify the validity of your final result: $\hat{f}(1) \approx -0.0645\mathrm{i}$, $\hat{f}(-8) = 0 = \hat{f}(42)$.)

$\hat{f}(0) = \square$ and $\hat{f}(k) = \square$ (for $k \neq 0$).

(b) Determine at which points f is represented by its Fourier series, i.e., for which $x \in \mathbb{R}$ does

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi \mathrm{i} k x} \,?$$

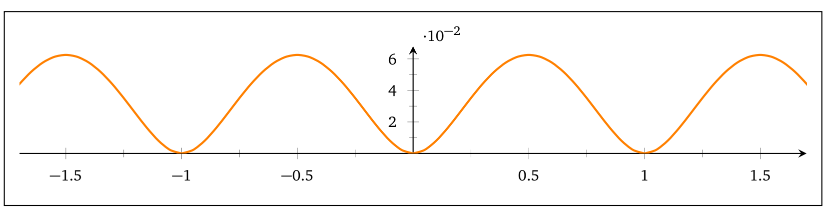
Answer: all $x \in \square$.

(c) Compute $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \pm \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \square$.

(Hint: approximations do not count. Use (b) together with a suitably chosen value for x. At the end, you should arrive at a formula for the quantity in question that you can comfortably enter into a calculator. The answer approximately equals 0.9689.)

10.3. *(Fourier series, III)* (4 credits)

Let $h\colon \mathbb{R} \to \mathbb{R}$ be the 1-periodic function defined by $h(x) = x^4 - 2x^3 + x^2$ for $0 \leq x < 1$:



(a) Compute the Fourier coefficients $\hat{h}(k)$ of h for $k \in \mathbb{Z}$.
(Hint: depending on how you go about doing this, this requires partial integration four times. You may check your final result using $\hat{h}(0) \approx 0.033$, $\hat{h}(1) \approx -0.015399$, $\hat{h}(1) \approx -0.015399$, $\hat{h}(-2) \approx -0.00096$.)

$\hat{h}(0) = \square$ and $\hat{h}(k) = \square$ (for $k \neq 0$).

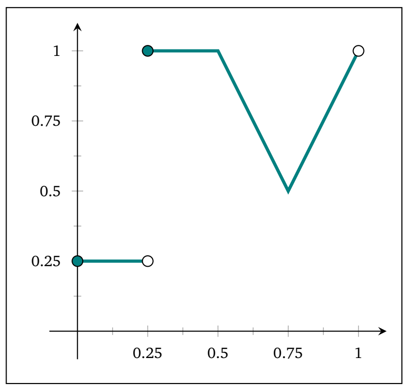
(b) Find complex numbers a_k and b_k such that

$$h(x) = \hat{h}(0) + \sum_{k=1}^{\infty} (a_k \cos(2\pi k x) + b_k \sin(2\pi k x))$$

holds for all $x \in \mathbb{R}$.
(Hint: exercise 3.2. Moreover, you can easily test your solution by replacing ∞ in the sum by 3 [the series in question converges rather quickly], plotting the resulting sum on $[0,1]$ and comparing with a plot of h. They should look almost identical.)

$a_k =$ $\boxed{}$ and $b_k =$ $\boxed{}$.

10.4. *(Fourier series, IV)* (4 credits)
Let $v\colon \mathbb{R} \to \mathbb{R}$ be the 1-periodic function whose graph on $[0,1]$ is given as follows:



Compute the Fourier coefficients $\hat{v}(k)$ of v for $k \in \mathbb{Z}$.

$\hat{v}(0) =$ $\boxed{}$ and $\hat{v}(k) =$ $\boxed{}$ (for $k \neq 0$).

(Hint: write v as a linear combination of functions whose Fourier coefficients you already know from § 4.3 of the lecture notes. By doing so, you should be able to just write down the correct answer without any computation.)